A STUDY ON THE EXISTENCE AND THE PROPERTIES
OF RANDOM IMPULSIVE DIFFERENTIAL SYSTEMS

Thesis submitted in Partial fulfilment for the award of
Degree of Doctor of Philosophy in Mathematics

Submitted by

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December - 2014
DECLARATION

I, M. GOWRISANKAR declare that the thesis entitled “A STUDY ON THE EXISTENCE AND THE PROPERTIES OF RANDOM IMPULSIVE DIFFERENTIAL SYSTEMS” submitted by me for the Degree of Doctor of Philosophy in Mathematics is the record work carried out by me during the period from January 2011 to December 2014 under the guidance of Dr. P. MOHANKUMAR M.Sc., M.Phil., Ph.D., Professor, Department of Mathematics, AVIT, Chennai, India and that this has not formed the basis for the award of any other degree, diploma, associate ship, fellowship, titles in this University or any other University or other similar institutions of higher learning.

Signature of the Candidate

Place: Salem
Date: 28.12.2014
I, Dr. P. MOHANKUMAR certify that the thesis entitled “A STUDY ON THE EXISTENCE AND THE PROPERTIES OF RANDOM IMPULSIVE DIFFERENTIAL SYSTEMS” submitted for the Degree of Doctor of Philosophy in Mathematics by M. GOWRISANKAR is the record of research work carried out by him during the period from January 2011 to December 2014 under my guidance and supervision and that this work has not formed the basis for the award of any degree, diploma, associate ship, fellowship or other titles in this University or any other University or Institution of higher learning.

Signature of the Supervisor
with designation

Place: Salem
Date: 28.12.2014
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Contents
<table>
<thead>
<tr>
<th>CHAPTERS</th>
<th>TITLES</th>
<th>Page No</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Impulsive Differential System</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Fixed Time Impulsive Events</td>
<td>5</td>
</tr>
<tr>
<td>1.3</td>
<td>Variable Time Impulsive Events</td>
<td>6</td>
</tr>
<tr>
<td>1.4</td>
<td>Discontinuous Dynamical Systems</td>
<td>8</td>
</tr>
<tr>
<td>1.5</td>
<td>Random Time Impulsive Events</td>
<td>9</td>
</tr>
<tr>
<td>1.6</td>
<td>Motivation</td>
<td>11</td>
</tr>
<tr>
<td>1.7</td>
<td>Stability Theory</td>
<td>18</td>
</tr>
<tr>
<td>1.7.1</td>
<td>Lyapunov Stability Theory</td>
<td>20</td>
</tr>
<tr>
<td>1.7.2</td>
<td>Ulam - Hyers -Rassias Stability</td>
<td>21</td>
</tr>
<tr>
<td>1.8</td>
<td>Existence, Stability and Methods</td>
<td>24</td>
</tr>
<tr>
<td>1.9</td>
<td>Preliminaries, Definitions and Lemma</td>
<td>28</td>
</tr>
<tr>
<td>1.10</td>
<td>Thesis Outline and Overview</td>
<td>33</td>
</tr>
<tr>
<td>1.11</td>
<td>Contributions of the Author</td>
<td>34</td>
</tr>
<tr>
<td>Chapter 2</td>
<td><strong>Existence and Stability Results on Nonlinear Delay Integro-Differential Equations with Random Impulses</strong></td>
<td>35</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>35</td>
</tr>
<tr>
<td>2.2</td>
<td>Preliminaries</td>
<td>37</td>
</tr>
<tr>
<td>2.3</td>
<td>Existence Results</td>
<td>40</td>
</tr>
<tr>
<td>2.4</td>
<td>Continuous Dependence</td>
<td>53</td>
</tr>
<tr>
<td>2.5</td>
<td>Ulam – Hyers - Rassias Type Stability</td>
<td>56</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>Stability Results of Random Impulsive Semilinear Differential Equations</td>
<td>64</td>
</tr>
<tr>
<td>-----------</td>
<td>-------------------------------------------------</td>
<td>----</td>
</tr>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>64</td>
</tr>
<tr>
<td>3.2</td>
<td>Preliminaries</td>
<td>67</td>
</tr>
<tr>
<td>3.3</td>
<td>Stability</td>
<td>74</td>
</tr>
<tr>
<td>3.4</td>
<td>Ulam - Hyers - Rassias Stability</td>
<td>77</td>
</tr>
<tr>
<td>3.5</td>
<td>Exponential Stability</td>
<td>84</td>
</tr>
<tr>
<td>3.6</td>
<td>Application</td>
<td>93</td>
</tr>
<tr>
<td>Chapter 4</td>
<td>Existences, Uniqueness and Stability of Random Impulsive Neutral Partial Differential Equations</td>
<td>97</td>
</tr>
<tr>
<td>4.1</td>
<td>Numerical Algorithm</td>
<td>97</td>
</tr>
<tr>
<td>4.2</td>
<td>Preliminaries</td>
<td>99</td>
</tr>
<tr>
<td>4.3</td>
<td>Existence and Uniqueness</td>
<td>104</td>
</tr>
<tr>
<td>4.4</td>
<td>Stability</td>
<td>108</td>
</tr>
<tr>
<td>4.5</td>
<td>An Example</td>
<td>111</td>
</tr>
<tr>
<td>Chapter 5</td>
<td>Existences, Uniqueness and Stability Results of Random Impulsive Fractional Differential Equations</td>
<td>115</td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
<td>115</td>
</tr>
<tr>
<td>5.2</td>
<td>Preliminaries</td>
<td>117</td>
</tr>
<tr>
<td>5.3</td>
<td>Existence and Uniqueness</td>
<td>121</td>
</tr>
<tr>
<td>5.4</td>
<td>Stability</td>
<td>124</td>
</tr>
<tr>
<td>5.5</td>
<td>Ulam - Hyers ′ ts</td>
<td>127</td>
</tr>
<tr>
<td>5.6</td>
<td>An Example</td>
<td>134</td>
</tr>
<tr>
<td>Chapter 6</td>
<td>Conclusions</td>
<td>139</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>140</td>
</tr>
<tr>
<td></td>
<td>List of Publications</td>
<td>155</td>
</tr>
</tbody>
</table>
Chapter - 1
1.1 Impulsive Differential System

When a real world problem is transformed into a deterministic initial value problem of ordinary differential equations, namely

$$\frac{dx}{dt} = f(t,x), \; x(t_0) = x_0$$

(1.1.1)

or a system of differential equations, cannot usually be sure that the model is perfect.

For example, the initial value may not be known exactly or, the function $f$ may contain uncertain parameters and systems that are subjected to rapid changes at certain instants in time. If they are estimated through certain measurements, they are necessarily subject to errors. The analysis of the effect of these errors leads to the study of the qualitative behavior of the solutions of (1.1.1). Significant interest in the investigation of systems with discontinuous trajectories is explained by the development of equipment in which significant role is played by impulsive differential systems and impulsive computing systems. Impulsive systems are also encountered in numerous problems of natural sciences described by mathematical models with conditions reflecting the
impulsive action of external forces with pulses whose duration can be neglected. It was discovered that the presence of a pulse action may significantly complicate the behavior of trajectories of these systems even in the case of quite simple differential equations.

There are numerous examples of evolutionary systems that are subjected to rapid changes at certain instants in time. In the simulations of such processes, it is frequently convenient and valid to neglect durations of the rapid changes and to assume that, the changes be represented by state jumps. Appropriate mathematical models for such processes are so-called differential systems with impulsive effects. It is now being recognized that the theory of impulsive differential systems is not only richer than the corresponding theory of differential systems but also represents a more natural framework for mathematical modeling of many real world phenomena.

The mathematical model of an evolving process with impulse effects is given by

1. A differential equation

\[
\frac{dx}{dt} = f(t, x), \quad (1.1.2)
\]

where \( x \in X \) and \( t \in \mathbb{R} \).

2. A switching set \( D_t \in X \times \mathbb{R} \).
3. A jumping operator $I_t$ defined on the switching set $D_t$ such that $I_t \circ D_t \subset X \times \mathbb{R}$.

The state variable $x(t)$ represents the state of the evolving process at time $t \in T$. With an initial state, $x(t_0^+) = x_0$, equation (1.1.2) defines a dynamical system which is the mathematical formalization of deterministic processes. The solution to (1.1.2) is often written as $\phi_t(x_0)$, or $x(t) = x(t, t_0, x_0)$. The map $\phi_t : X \to X$ is called the evolution operator of the dynamical system, where $X$ is a metric space. The one-parameter family of mappings $\{\phi_t\}_{t \in T}$ satisfies

$$\phi_{t+s} = \phi_t \circ \phi_s$$

and

$$\phi_0(x) = x,$$

is called the flow. The set of points $\{\phi_t(x_0) / t \in T\}$ is called a trajectory, or an orbit of equation (1.1.2) through $x_0$.

If an impulsive event happens at $t = \tau$, then the trajectory hits the switching set $D_t$ at $t = \tau$; namely, $x(\tau) \in D_t$. As soon as $x(t)$ hits the switching set $D_t$ at $t = \tau$, it "jumps" immediately to a point defined by the jumping operator $I_t$; namely, $x(\tau^+) = I_t \circ x(\tau)$. Thus the solution of the impulsive system $\phi_t(x_0)$, satisfies the equation (1.1.2) outside the
switching set $D_i$ and is discontinuous of the first kind at the points where it hits the switching set $D_i$ with the jump

$$\Delta x(t) = \phi_i^{-1}(x_0) - \phi_i(x_0) = \mathbb{1}_i \circ \phi_i(x_0) - \phi_i(x_0). \quad (1.1.3)$$

then there is a concise form of impulsive differential equation as

$$\frac{dx}{dt} = f(t,x), \quad (t,x) \not\in D_i,$$

$$\Delta x = \mathbb{1}_i \circ x - x, \quad (t,x) \in D_i. \quad (1.1.4)$$

Equation (1.1.4) can have the following three kinds of solutions:

(i) $\phi_i(x_0)$ does not hit $D_i$ or the hitting points are fixed points of $\mathbb{1}_i$. In this case, there is no impulsive event.

(ii) $\phi_i(x_0)$ hits $D_i$ at finite number of points that are not fixed points of $\mathbb{1}_i$. In this case, there are finite number of impulsive events.

(iii) $\phi_i(x_0)$ hits $D_i$ in a countable number of points that are not fixed points of $\mathbb{1}_i$. In this case, there are countable number of impulsive events.

Based on different characteristics of impulsive events, usually four primary types of impulsive systems are encountered

1. in which impulses occur at fixed time;
2. in which impulses occur when the trajectory hits the hyper surface in the extended phase space \((X \times \mathbb{R})\);

3. which are discontinuous dynamical systems;

4. in which impulses occur at random time.

### 1.2 Fixed Time Impulsive Events

If the impulsive events occur in a finite or infinite sequence of time \(\{\tau_k\}\), then system (1.1.4) can be written as

\[
\begin{align*}
\frac{dx}{dt} &= f(t,x), \quad t \neq \tau_k, \\
\Delta x &= I_k, \quad t = \tau_k. 
\end{align*}
\]  

(1.2.1)

The solution of equation (1.2.1) is a piecewise continuous function \(x = \phi(t)\) that has discontinuities of the first kind at \(t = \tau_k\).

\[
\frac{d\phi(t)}{dt} = f(t,\phi(t)) \text{ is satisfied for all } t \neq \tau_k. \quad \text{At } t = \tau_k, \phi(t) \text{ satisfies the following jumping condition:}
\]

\[
\Delta \phi|_{t=\tau_k} = \phi(\tau_k^+) - \phi(\tau_k^-) = I_k(\phi(\tau_k^+)).
\]

(1.2.2)

In some references, equation (1.2.2) is written as

\[
\Delta \phi|_{t=\tau_k} = \phi(\tau_k^+) - \phi(\tau_k^-) = I_k(\phi(\tau_k^-)).
\]

(1.2.3)
For the basic monographs for fixed time impulsive events refer [10,11,46, 67,72] and articles [5,16,18,19,20,28,30,33,34,36,61,62] for numerous examples and further discussions.

1.3 Variable Time Impulsive Events

In this type of system, the impulsive events occur when trajectories hit a hypersurface $\mathcal{H}(t,x) = 0$. This kind of impulsive differential system can be written as

$$\frac{dx}{dt} = f(t,x), \quad \mathcal{H}(t,x) \neq 0,$$

$$\Delta x = I(t,x), \quad \mathcal{H}(t,x) = 0. \quad (1.3.1)$$

In this case, the switching set is given by

$$D_i = \{(t,x) | \mathcal{H}(t,x) = 0\}, \quad (1.3.2)$$

and the jump operators $I_i$ is given by

$$I_i : (t,x) \rightarrow (t,x + I(t,x)) \quad (1.3.3)$$

If the equation $\mathcal{H}(t,x) = 0$ has a countable number of solutions with respect to $t$, denoting these solutions by $t = \tau_k(x)$ and index then by the set of integers (or a subset of integers) such that $\tau_k(x) \rightarrow \infty$ as $k \rightarrow \infty$ and $\tau_k(x) \rightarrow -\infty$ as $k \rightarrow -\infty$. In this case, the jumping operator is given by
which can be simplified as

\[ l_{\tau_k(x)} : x \rightarrow x + I(\tau_k(x), x), \quad (1.3.4) \]

Then equation (1.3.1) can be rewritten as

\[ l_{\tau_k(x)} : x \rightarrow x + I_k(x). \quad (1.3.5) \]

\[
\frac{dx}{dt} = f(t, x), \quad t \neq \tau_k(x),
\]

\[
\Delta x = I_k(x), \quad t = \tau_k(x). \quad (1.3.6)
\]

System (1.3.1) is more difficult to study than system (1.2.1). The solutions of system (1.3.1) starting at different initial conditions have different points of discontinuity. A solution of system (1.3.1) may hit the same hypersurface \( \mathcal{H}(t, x) = 0 \) many times and cause so called “beating phenomenon” or “pulse phenomenon”. Different solutions of system (1.3.1) may coincide after some time and behave like a single solution thereafter and thus cause “confluence”.

There are several papers which study the impulsive differential systems with variable times. In [43, 47], V. Lakshmikantham, S. Leela and S. Kaul., discussed existence and stability criteria by using comparison principle and extremal solution for the impulsive differential equations with variable times. D.D. Bainov and I.M. Stamova [12] and P.
S. Simeonov and D.D. Bainov [69] investigated the stability of the solutions of equations with variable time impulses. Ignacio Bajo [38] studied the pulse accumulation in first order impulsive differential equation and gave necessary and sufficient conditions to ensure pulse accumulation in such equations. The monographs [46, 67, 72] give a variety of treatment of the notion of “differential equations with variable times”.

1.4 Discontinuous Dynamical Systems

When the differential system (1.1.2) does not explicitly depend on $t$ and the jumping operator $l_t = l$ for all $t \in \mathbb{R}$, let $l : D \rightarrow D_0$ be a mapping between the switching set $D$ and a target set $D_0$, then a discontinuous dynamical system is given by

$$\frac{dx}{dt} = f(x), \quad x \notin D,$$

$$\Delta x = l(x) - x = I(x), \quad x \in D.$$

(1.4.1)

Many electrical circuit models containing switching capacitors belong to this kind of impulsive system. For further study of discontinuous dynamical systems refer the fundamental monographs in [29, 72].
1.5 Random Time Impulsive Events

Let \( \{\tau_i, i = 1, 2, \ldots\} \) be a series of random variable and \( \tau_i \in D_i = (0, d_k) \), where \( 0 < d_k \leq \infty \), \( \tau_i \) is independent of \( \tau_j \) when \( i \neq j \) for all \( i, j = 1, 2, \ldots; \) then the system (1.1.4) becomes

\[
\frac{dx}{dt} = f(t, x), \quad a.e., t \neq \tau_k,
\]

\[
\Delta x = I_k(\tau_k, x(\tau_k)), \quad a.e., t = \tau_k, k = 1, 2, \ldots, \quad (1.5.1)
\]

where \( \tau_k \) is the \( k^{th} \) impulse moment, which is random variable. When the impulses exist at random, the solutions of the equation behave as a stochastic process. It is quite different from deterministic impulsive system and stochastic differential system.

(RIDE) and investigated boundedness of solutions to these models by Liapunov’s direct method. In [83] S.J. Wu investigated the Euler scheme for RIDEs which is an important application to generate the whole approximate trajectories of RIDEs. S.J. Wu, X.L. Guo et al [85,86,87] have studied the qualitative properties of differential equations with random impulses.

Recently, the interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With this advantage, the fractional order models become more realistic and practical than the classical integer order models, in which such effects are not taken into account. As a matter of fact, fractional differential equations (FDEs) arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics and fitting of experimental data. Fractional differential equations involving the Riemann -Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention For more recent development on this hot topic, one can see the monographs of A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo [44] and I. Podlubny [58].
In this thesis, the system having random time impulses is considered.

1.6 Motivation

The object of this thesis is to find the existence and stability of solutions of various types of random time impulsive differential equations. Here listed out some examples that motivate the investigate of impulsive differential systems.

1.6.1 Example [70]

Most of the models of single species dynamics have been derived from differential equations of the form

\[ \dot{x} = x f(t,x) + g(t,x), \]  \hspace{1cm} (1.6.1)

where the solution \( x = x(t) \) is treated as the population size (or biomass) in time \( t > 0 \), the function \( f = f(t,x) \) is characterized by the population change at the moment \( t \), the function \( g = g(t,x) \) describes the continuous influence of out-side factors. Various choices of the functions \( f \) and \( g \) lead us to the various differential equation models. For instance:

1. If \( f(t,x) = \frac{a}{b} (b - x) \), the Verhulst differential is equation obtained.
\[ \dot{x} = \frac{a}{b} x(b - x) + g(t, x), \]  

(1.6.2)

where \( a = a(t) \in \mathbb{R} = [0, \infty) \) is the reproductive potential of population;

\( b = b(t) \in \mathbb{R}_+ \) is the capacity of environment.

2. If \( f(t, x) = a - c \ln x \), the Gompertz differential equation is obtained.

\[ \dot{x} = x(a - c \ln x) + g(t, x), \]  

(1.6.3)

where \( c = c(t) \in \mathbb{R}_+ \) is the coefficient of inter species competition.

3. If \( f(t, x) = f(x) \), then equation (1.6.1) is an evolutionary model of stationary population.

4. If \( g(t, x) = 0 \), then equation (1.6.1) is an evolutionary model of isolated population.

Let \( \eta = \eta(t; t_0, x_0) \) be a solution of the differential equation (1.6.1) with initial condition

\[ \eta(t_0; t_0, x_0) = x_0, \quad (t_0, x_0) \in \mathbb{R}_+^2. \]  

(1.6.4)

Let \( \tau_1 < \cdots < \tau_p \), \( t_0 < \tau_1 \), \( p \in \mathbb{N} = \{1, 2, \cdots\} \) be the moments of outside perturbations on the evolution of the considered population system, for example, subtracting or adding some quantity of biomass, etc., then
\[ x(\tau_i + 0; t_0, x_0) = \Phi(\tau_i, x(\tau_i; t_0, x_0)), \quad i \in \mathbb{N}_p = \{1, 2, \ldots, p\}, \quad (1.6.5) \]

where \( x(\tau_i + 0; t_0, x_0) = \lim\{x(t; t_0, x_0) : t \rightarrow \tau_i, t > \tau_i\} \); \( \Phi = \Phi(t, x) \) is a map, which characterized the outside action in the moment \( t = \tau_i \). For example, if \( \Phi(\tau_i, x) = x - d_i, \quad d_i > 0, \quad i \in \mathbb{N}_p \) then in each moment \( \tau_i \), subtract the quantity \( d_i \) from the population biomass. The system (1.6.1) and (1.6.4) together with equation (1.6.5) is called a \textit{fixed type impulsive system}.

\textbf{1.6.2 Example [26]}

\textbf{Impulsive System in Fed-Batch Culture of Fermentative Production:}

The fed-batch culture of glycerol bioconversion to 1,3-PD begins with batch fermentation, then batch-fed glycerol and alkali is added to the reactor in order that the glycerol concentration keeps in a given range and pH value in the bioreactor maintains 7 or so. At the end of fermentation, glycerol is expected to convert to 1,3-PD as much as it can. The whole process includes batch fermentation in the early stage and later fed-batch culture. Mass balance of biomass, substrate and products in batch microbial cultures are written as follows:
\[
\begin{align*}
\dot{x}_1(t) &= \mu x_1(t), \\
\dot{x}_2(t) &= -q_2 x_3(t), \\
\dot{x}_3(t) &= q_3 x_1(t), \quad t \in [0, t_1), \\
\dot{x}_4(t) &= q_4 x_1(t), \\
\dot{x}_5(t) &= q_5 x_1(t),
\end{align*}
\tag{1.6.6}
\]

where \(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)\) are respectively, biomass, glycerol, 1,3-PD, acetate and ethanol concentrations at \(t\) in reactor, \(\xi \in \mathbb{R}^5\) is the initial state. The specific growth rate of cells \(\mu\), specific consumption rate of substrate \(q_2\) and specific formation rate of product \(q_i, i = 3, 4, 5\), are expressed by the following equations on the basis of work in [89]:

\[
\begin{align*}
\mu &= \mu_m \frac{x_2(t)}{x_2(t) + k_s \prod_{i=2}^5 (1 - \frac{x_i(t)}{x_i^*})}, \\
q_2 &= m_2 + \mu \frac{x_2(t)}{Y_2}, \\
q_3 &= m_3 + \mu Y_3 + \Delta_3 \frac{x_2(t)}{x_2(t) + k_3}, \\
q_4 &= m_4 + \mu Y_4 + \Delta_4 \frac{x_2(t)}{x_2(t) + k_4}, \\
q_5 &= q_2 \left( \frac{b_1}{c_1 + \mu x_2(t)} \right) + \frac{b_2}{c_2 + \mu x_2(t)}.
\end{align*}
\]

Under anaerobic conditions at 37\(^\circ\) C and \(pH\) 7.0, the maximum specific growth rate \(\mu_m\) and Monod constant \(k_s\) are 0.67\(h^{-1}\) and 0.28 mmol / L respectively. And \(b_1, b_2, c_1, c_2, m_i, Y_i, \Delta_i, k_i, i = 2, 3, 4\) are parameters according to the work in [89]. The critical concentration of biomass, glycerol, 1,3-PD, acetate and ethanol for cell growth are
\[ x_1^* = 10 \text{ g} / \text{L} , \ x_2^* = 2039 \text{ mmol} / \text{L} , \ x_3^* = 939.5 \text{ mmol} / \text{L} , \]
\[ x_4^* = 1024 \text{ mmol} / \text{L} \text{ and } x_5^* = 306.9 \text{ mmol} / \text{L} \text{ respectively.} \]

Since biomass, glycerol and products cannot exceed their critical concentrations according to the practical production, the properties of the system are considered on the subset of \( \mathbb{R}^5 \)

\[ W := \{ x \in \mathbb{R}^5 / x_1 \in [0.001, x_1^*], x_2 \in [100, x_2^*], x_i \in [0, x_i], i = 3, 4, 5 \} . \]

In order to decrease the influence of the inhibition of substrates and multi-products, alkali is added to the reactor while adding glycerol to neutralize the product acetate, which has little effect on the productivity of 1,3-PD. The effects of that are ignored in the dynamical system. Consequently assume the following assumptions:

\( (H_1) \): The glycerol concentration is uniform in reactor while adding glycerol to the reactor, time delay and nonuniform space distribution are ignored.

\( (H_2) \): The feed rate of glycerol can be infinitely large. Consequently, it is possible to add the substrate instantaneously to the reactor at various discrete time instants.

Under assumption \( (H_1) \) and \( (H_2) \), the impulsive system of fed-batch culture can be obtained.
Let \( x(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)) \in \mathbb{R}^5 \) be the state variable, \( I_1 = [0, t_1) \) be the interval of batch fermentation, \( I_2 = [t_1, T] \) be the interval of fed-batch fermentation,

\[
I = I_1 \cup I_2, \quad D = \{t_1, t_2, \ldots, t_n\} \subset [t_1, T), \quad t_i \in I_n = \{1, 2, \ldots, n\},
\]

where \( t_i \) are the moments when adding glycerol, \( t_n = T \).

Thus, the fed-batch culture including batch fermentation can be formulated as the following nonlinear impulsive system:

\[
\dot{x}(t) = f(x(t)), \quad t \in I \setminus D, x(0) = \xi,
\]

\[
\Delta x(t_i) = G_i(x(t_i)), \quad 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T,
\]

where \( f(x(t)) = (\mu x_1(t), -q_2 x_1(t), q_3 x_1(t), q_4 x_1(t), q_5 x_1(t)) \in \mathbb{R}^5 \),

\[
G_i(x(t_i)) = (-u_i x_1(t_i), -u_i x_2(t_i) + cu_i, -u_i x_3(t_i), -u_i x_4(t_i), -u_i x_5(t_i)) \in \mathbb{R}^5,
\]

\( \xi \in W \) is the initial state, \( \Delta x(t_i) = x(t_i^+) - x(t_i^-) = x(t_i^+) - x(t_i^-) \) is the jump size in the states at \( t_i, i \in I_n \). \( c \) is the initial glycerol concentration in feed. \( u_i = F_i / (\sum_{k=1}^{i} F_k + V) \) is the dilution rate at \( t_i \), where \( F_i \) is the volume of glycerol added at \( t_i, i = 1, 2, \ldots, n \). \( V \) is the initial volume of fermentation broth.
1.6.3 Example [83]

Interest Rate Model

The time when interest rate is adjusted, is a random variable. However, interest rate is a constant during two neighboring adjusted times. Thus, the interest rate \( r(t) \) can be modeled by random impulsive differential equation

\[
\begin{aligned}
\frac{dr(t)}{dt} &= 0, \quad t \neq t_k, \\
\quad r(t_k) &= I_k(r(t_k)), \quad k = 1, 2, \ldots,
\end{aligned}
\]

where \( t_k \) denote the times that the interest rate is adjusted, which are a series of random variables. \( I_k \) is some pending function of \( r(t_k) \).

1.6.4 Example [39]

The Response of a Duffing Oscillator

As an example of a system with non-linear stiffness, a Duffing oscillator is to be considered here. Such a problem typically occurs in idealizing a structure as a single degree of freedom system when the non-linearity due to large deflections is taken into account (see the example of an elastic plate with large deflections in reference [50]).

The response of a Duffing oscillator to a Poisson-distributed train of impulses is to be considered, governed by the equation
The excitation is a train of Dirac delta impulses $\delta(t-t_i)$ occurring at random times $t_i$. It is assumed that the occurrences of the impulses are Poisson distributed and that $N(t)$ is a counting process giving the number of impulses in the time interval $[0,t)$ with the initial condition $N(0) = 0$, with probability one. The present definition of the counting process as the number of arrivals in the closed-at-the-lower-end and open-at-the-upper-end interval, which is an alternative with respect to the usual one, has been used. The impulses magnitudes $P_i$ are assumed to be independent random variables, identically distributed as a random variable $P$. The Poisson process is completely characterized by the expected rate $\nu(t)$ of events (impulse occurrences). From the regularity assumption about the counting process $N(t)$, it follows that

$$E[N(dt)] \approx \nu(t)dt + O(dt),$$

(1.6.10)

where $N(dt)$ denotes the random number of impulses in the infinitesimal time interval $[t,t+dt)$.

### 1.7 Stability Theory

In 1892, Lyapunov introduced the concept of stability of a dynamic system. Roughly speaking, stability means insensitivity of the state of the
system to small changes in the initial state or the parameters of the system. For a stable system, the trajectories which are "close" to each other at a specific instant should therefore remain close to each other at all subsequent instants. Many parts of the qualitative theory of differential equations and dynamical systems deal with asymptotic properties of solutions and the trajectories. The simplest kind of behavior is exhibited by equilibrium points, or fixed points, and by periodic orbits. If a particular orbit is well understood, it is natural to ask whether a small change in the initial condition will lead to similar behavior. Stability means that the trajectories do not change too much under small perturbations. The opposite situation, where a nearby orbit is getting repelled from the given orbit, is also of interest. In general, perturbing the initial state in some directions results in the trajectory asymptotically approaching the given one and in other directions to the trajectory getting away from it. There may also be directions for which the behavior of the perturbed orbit is more complicated (neither converging nor escaping completely), and then stability theory does not give sufficient information about the dynamics. One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit.
Concept of stability in common and Engineering sense reflects necessity to keep response of a disturbed system within accepted limits. If deviations describing response of the system from a given regime (e.g., state of equilibrium) lie within prescribed limits, the system is called stable. Otherwise, the system is called unstable. Disturbances, response and prescribed limits can be specified in each case in different ways.

1.7.1 Lyapunov Stability Theory

In Mathematics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. Lyapunov developed the stability theory of dynamical systems determined by nonlinear time-varying ODEs. In the concepts of stability and instability, he has developed two general methods for the stability analysis of an equilibrium: Lyapunov’s direct method, is called the second method while his indirect method is also called the first method. The former involves the existence of scalar-valued auxiliary functions of the state space (called Lyapunov functions) to ascertain the stability properties of an equilibrium, whereas the latter seeks to deduce the stability properties of an equilibrium of a system described by a nonlinear DE from the stability properties of its linearization. In the process of discovering the first method, Lyapunov
established some important stability results for linear systems (involving the Lyapunov matrix equation).

An orbit is called Lyapunov Stable if the forward orbit of any point in a small enough neighborhood of it stays in a small (but perhaps, larger) neighborhood. Various criteria have been developed to prove stability or instability of an orbit. Under favorable circumstances, the problem of stability may be reduced to a well-studied problem involving eigenvalues of matrices. A more general method involves Lyapunov functions where as Lyapunov stability theorems give only sufficient condition.

Lyapunov method is a powerful method for determining the stability or instability of fixed points of nonlinear autonomous systems. In mathematics, Lyapunov functions are functions which can be used to prove the stability of a certain fixed point in a dynamical system.

1.7.2 Ulam - Hyers -Rassias Stability

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping?. The first answer to the question of Ulam was given by Hyers in 1941 in the case of Banach spaces (See [37, 59, 73]).
Let $E_1, E_2$ be two real Banach spaces and $\varepsilon > 0$. Then for every mapping $f : E_1 \to E_2$ satisfying

$$P_f(x + y)f(x)f(y) \leq \varepsilon \quad \text{for all } x, y \in E_1.$$ 

There exists a unique additive mapping $g : E_1 \to E_2$ with the property

$$P_f(x)g(x) \leq \varepsilon, \quad \text{for all } x \in E_1.$$ 

Thereafter, this type of stability is called the Ulam-Hyers stability.

In 1978, Rassias provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables.

Consider the following random impulsive functional differential equations:

$$\begin{align*}
x'(t) &= f(t, x_t), \quad t \neq \xi_k, \quad t \geq t_0 \\
x(\xi_k) &= I_k (\xi_k, x(\xi_k)), \quad k \in N \\
x(t_0) &= \varphi.
\end{align*}$$

(1.7.1)

**Ulam-Hyers Stability**

For studying the Ulam-Hyers stability of random impulsive fractional differential equation (1.7.1). Let $\varepsilon > 0$ and $\phi : [\tau, T] \to \mathbb{R}^+$ be a continuous function. Consider the following inequalities:

$$\begin{align*}
P_f(x(t)) - f(t, x_t)^P &\leq \varepsilon, \quad t \neq \xi_k, \quad t \geq \tau. \\
P_0(x(\xi_k)) - b_k(\tau_k) x(\xi_k)^P &\leq \varepsilon, \quad k = 1, 2, \ldots.
\end{align*}$$

(1.7.2)
Definition 1.7.1

The system (1.7.1) is Ulam-Hyers stable in the mean square if there exists a real number \( \kappa > 0 \) such that for each \( \varepsilon > 0 \) and for each solution \( x \in \mathcal{B} \) of the inequality (1.7.2) there exists a solution \( y \in \mathcal{B} \) of the system (1.7.1) with \( E \| P x(t) - y(t) \|^2 \leq \kappa \varepsilon \), \( t \in [\tau, T] \).

Definition 1.7.2

The system (1.7.1) is generalized Ulam-Hyers stable in the mean square if there exists a real number \( \eta \in \mathcal{B}, \eta(0) = 0 \) such that for each solution \( x \in \mathcal{B} \) of the inequality (1.7.2) there exists a solution \( y \in \mathcal{B} \) of the system (1.7.1) with \( E \| P x(t) - y(t) \|^2 \leq \eta(\varepsilon) \), \( t \in [\tau, T] \).

Definition 1.7.3

The system (1.7.1) is Ulam-Hyers-Rassias stable in the mean square with respect to \( \phi \) if there exists a real number \( \zeta > 0 \) such that for each \( \varepsilon > 0 \) and for each solution \( x \in \mathcal{B} \) of the inequality (1.7.4) there exists a solution \( y \in \mathcal{B} \) of the system (1.7.1) with

\[
E \| P x(t) - y(t) \|^2 \leq \zeta \varepsilon \phi(t), t \in [\tau, T].
\]
Definition 1.7.4

The system (1.7.1) is generalized Ulam- Hyers- Rassias stable in the mean square with respect to $\phi$ if there exists a real number $\zeta > 0$ such that for each solution $x \in B$ of the inequality (1.7.3) there exists a solution $y \in B$ of the system (1.7.1) with $E \| x(t) - y(t) \|^2 \leq \zeta \phi(t)$, $t \in [\tau, T]$.

1.8 Existence, Stability and Methods

In recent years, impulsive differential equations have been intensively researched (see monographs [10,11,29,46,67,72]). The theory of differential equations with impulse effect has been the subject of many investigations due to the wide application of these system to the control theory, biology, electronics, etc. The effective use of systems with impulse effect in mathematical modeling of various processes and phenomena requires the formulation of effective criteria for stability of their solutions. Now there also exists a well developed qualitative theory of impulsive (fixed times and other types) differential equations. However, not so much has been developed in the direction of random impulsive differential equations (RIDEs) and fixed type impulsive stochastic differential equations (ISDEs).

Till now only few papers have been published on the topic. In the few publications dedicated to this subject, earlier works done by R.

The problems on qualitative properties of RIDEs have been investigated by S.J. Wu and X.Z. Meng [84], first brought forward random impulsive ordinary differential equations and investigated boundedness of solutions to these models by the Liapunov’s direct method. In [85], S.J. Wu and Y.R. Duan discussed oscillation, stability and boundedness of solutions to the model by comparing the solutions of this system with the corresponding non-impulsive differential system. In [87], S.J. Wu, X.L. Guo and Y. Zhou first brought forward random impulsive functional differential equations and considered $p$–moment stability of solutions to these models using Liapunov’s function coupled with Razumikhin technique. S.J. Wu [83], presented algorithm analysis of
Euler scheme for random impulsive differential equations and discussed its continuous dependence on initial values. S.J.Wu, X.L. Guo and R.H. Zhai [88], considered almost the sure stability of solutions to random impulsive functional differential equations by Liapunov’s function coupled with Razumikhin technique. Dianli Zhao and Lei Zhang [23], presented the exponential asymptotic stability of nonlinear Volterra equations with random impulses.

The standard exponential stability problem is to find sufficient conditions such that the solution \( x(t,x_0) \) initiated at \( x(0)=x_0 \) of the system satisfies the condition

\[
\exists \ N > 0, \alpha > 0; \ P x(t) \leq P x_0 P N e^{-\alpha t}, \forall t \geq 0.
\]

The positive number \( \alpha \) is the convergence rate. There are many different methods given to deal with the exponential stability problem. Among the well known Lyapunov stability methods, this method is a powerful tool for studying system stability. Yet, numerous difficulties with the theory and application to specific problems persist and it does seem that new methods are needed to address those difficulties.

Recently, T.A.Burton and his co-author (see monograph [14]), have applied fixed point theory to investigate the stability for deterministic systems, which show that some of these difficulties vanish when applying fixed point theory. J.A.D. Appleby [9] and Jiaowan Luo [40, 41] have used the fixed point theory to deal with the stability for
stochastic differential equations (SDEs). More precisely, J.A.D. Appleby [9], studied the almost sure stability for a classical equation by splitting the SDE into two equations, one being a fixed stochastic problem and the other a deterministic stability problem with forcing function. Jiaowan Luo [40, 41], made a different method than in [9] to investigate the exponential stability and asymptotical stability.

The deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore move from deterministic problems to stochastic problems. Stochastic differential equations have been greatly developed and they play an important role in many ways such as pricing an option, forecasting the growth of population and determining optimal portfolio of investments, etc. As the generalization of classical impulsive differential equations, impulsive stochastic differential equations have attracted the researchers’ great interest. There are few publications in the theory of ISDEs, Jun Yang, Shouming Zhong and Wenpin Luo [42], studied the stability analysis of impulsive stochastic differential equations with delays. Z. Yang, D. Xu and Li Xiang [90], studied the exponential p-stability of impulsive stochastic differential equations with delays. In [64, 65], R. Sakthivel and J. Luo studied the existence and asymptotic stability in $p^{th}$ moment of mild solutions to impulsive stochastic partial differential equations with and without infinite delays through fixed point theory. In [8], A. Anguraj, S.
Wu and A. Vinodkumar studied the existence and exponential stability for a random impulsive semilinear functional differential equations through the fixed point technique under non-uniqueness. The existence, uniqueness and stability results were discussed in A. Anguraj and A. Vinodkumar [6] through Banach fixed point method for the system of differential equations with random impulsive effect. A. Anguraj and A. Vinodkumar [7], studied the existence results for the random impulsive neutral functional differential equations with delays. In [74, 75], A. Vinodkumar and A. Anguraj studied existence results of random impulsive differential inclusions with delays via fixed point theory. In [92], S. Zhang and J. Sun generalized the distribution of random impulses with the Erlang distribution.

1.9 Preliminaries, Definitions and Lemmas

In this section, some useful definitions and lemmas are stated below which are useful to derive the main results in the following chapters.

**Definition 1.9.1**

*All the possible outcomes, the elementary events are grouped together to form a set $\Omega$ with typical element $\omega \in \Omega$. Not every subset of $\Omega$ is in general an observable or interesting event. So group only these observable or interesting events together as a family $F$ of subsets of $\Omega$.***
For the purpose of probability theory, such a family $F$ should have the following properties:

- $\emptyset \in F$, where $\emptyset$ denotes the empty set,
- $\alpha \in F \Rightarrow \alpha^C \in F$, where $\alpha^C = \Omega - \alpha$ is the complement of $\alpha$ in $\Omega$,
- $\{\alpha_i\}_{i \geq 1} \subseteq F \Rightarrow \bigcup_{i=1}^{\infty} \alpha_i \in F$.

A family $F$ with these three properties is called a $\sigma$-algebra. The pair $(\Omega, F)$ is called a measurable space, and elements of $F$ is henceforth called $F$–measurable sets instead of events.

**Definition 1.9.2**

A real valued function $x: \Omega \to \mathbb{R}$ is said to be $F$–measurable if $\{\omega : x(\omega) \leq a\}$ for all $a \in \mathbb{R}$. The function $x$ is also called a real–valued ($F$–measurable) random variable.

**Definition 1.9.3**

A probability measure $P$ on a measurable space $(\Omega, F)$ is a function $P: F \to [0,1]$ such that

- $P(\Omega) = 1$;
- for any disjoint sequence $\{\alpha_i\}_{i \geq 1} \subseteq F$ (that is $\alpha_i \bigcap \alpha_j = \emptyset$ if $i \neq j$)

$$P\left( \bigcup_{i=1}^{\infty} \alpha_i \right) = \sum_{i=1}^{\infty} P(\alpha_i).$$

The triple $(\Omega, F, P)$ is called a probability space.
**Definition 1.9.4**

If $(\Omega,F,P)$ is a probability space, it is set as

$$F = \{ \alpha \subseteq \Omega : \exists \beta, \gamma \in F \text{ such that } \beta \subseteq \alpha \subseteq \gamma, \quad P(\beta) = P(\gamma) \}.$$  

Then $F$ is a $\sigma$-algebra and is called the completion of $F$. If $F = \overline{F}$, the probability space $(\Omega,F,P)$ is said to be complete.

**Definition 1.9.5 (Stochastic processes)**

Families of random variables which are functions of time, are known as stochastic processes (or random processes or random functions).

Let $(\Omega,F,P)$ be a probability space. A filtration is a family $\{F_t\}_{t \geq 0}$ of increasing sub $\sigma$-algebras of $F$ (that is $F_t \subseteq F_s \subseteq F$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $F_t = \bigcap_{s > t} F_s$ for all $t \geq 0$. When the probability space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $F_0$ contains all $P$-null sets.

From now on, unless otherwise specified, it is always worked on a given complete probability space $(\Omega,F,P)$ with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions.
A family \( \{x(t)\}_{t \in I} \) of \( \mathbb{R}^d \)–valued random variables is called a **stochastic process** with parameter set (or index set) \( I \) and state space \( \mathbb{R}^d \). The parameter set \( I \) is usually the halfline \( \mathbb{R}_+ = [0, \infty) \), but it may also be an interval \([a, b] \), the nonnegative integers or even subsets of \( \mathbb{R}^d \). Note that for each fixed \( t \in I \), there is a random variable \( \omega \to x(t, \omega) \in \mathbb{R}^d \) for every \( \omega \in \Omega \).

On the other hand, for each fixed \( \omega \in \Omega \), there is a function

\[
t \to x(t, \omega) \in \mathbb{R}^d \quad \text{for every } t \in I
\]

which is called a sample path of the process.

Let \( \{x(t)\}_{t \geq 0} \) be an \( \mathbb{R}^d \)–valued stochastic process. It is said to be \( \{F_t\} \)–adapted (or simply, adapted) if for every \( t \), \( x(t) \) is \( F_t \)–measurable.

**Definition 1.9.6**

*The solution \( x = 0 \) of the system described by the equation*

\[
\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0,
\]  
where \( x \) and \( x_0 \) are elements of \( \mathbb{R}^n \) is said to be **stable** if for every \( \varepsilon > 0 \), there exist a \( \delta(t_0, \varepsilon) > 0 \) such that \( P_{x(t_0)}P < \delta \) implies \( P_{x(t)}P < \varepsilon \), for every \( t \geq t_0 \).
**Definition 1.9.7**

The solution \( x = 0 \) of the system (1.9.1) is said to be **asymptotically stable** if it is stable and if there exists a \( \delta(t_0) > 0 \) such that \( P x(t_0) \| < \delta \) implies \( \lim_{t \to \infty} x(t) = 0 \).

**Definition 1.9.8**

The solution \( x(0) = x_0 \) of the system (1.9.1) is said to be **exponential stability** if it is stable and if there exists \( N > 0, \alpha > 0 \) such that \( P x(t) \| \leq P x_0 \| e^{-\alpha t} \), \( \forall t \geq 0 \). The positive number \( \alpha \) is the convergence rate.

**Definition 1.9.9 (Banach Fixed Point Theorem)**

If \( X \) is a Banach space and \( T : X \to X \) is a contraction mapping then \( T \) has a unique fixed point.

**Definition 1.9.10 (Leray-Schauder alternative Fixed Point Theorem)**

Let \( B \) be a convex subset of a Banach space \( E \) and assume that \( 0 \in B \). Let \( F : B \to B \) be a completely continuous operator and let \( U(F) = \{ x \in B : x = \lambda Fx \text{ for some } 0 < \lambda < 1 \} \); then either \( U(F) \) is unbounded or \( F \) has a fixed point.
1.10 Thesis Outline and Overview

In this thesis the above mentioned techniques are used to prove the existence and stability results.

In chapter-2, the existence, uniqueness, stability via continuous dependence and Ulam stabilities of nonlinear integro-differential equations with random impulses are found out under sufficient conditions. The results are obtained by using Leray-Schauder alternative fixed point theorem and Banach contraction principle.

In chapter-3, the existence, uniqueness, continuous dependence, Ulam stabilities and exponential stability of random impulsive semilinear differential equations under sufficient conditions are proved. The results are obtained by using the contraction mapping principle. Finally an example is given to illustrate the applications of the abstract results.

In chapter-4, the existence, uniqueness and stability via continuous dependence of mild solution of neutral partial differential equations with random impulses are investigated under sufficient condition via fixed point theory.

In chapter-5, the existence, uniqueness, stability through continuous dependence on initial conditions and Hyers-Ulam-Rassias stability results for random impulsive fractional differential systems by
relaxing the linear growth conditions are given. Finally examples are provided to illustrate the applications of the abstract results.

Chapter-6, the objective results are discussed.

1.11 Contributions of the Author

In the light of the above, the author has obtained some significant generalizations on the following topics:

1. Existence and stability results on nonlinear delay integro-differential equations with random impulses.

2. Stability results of random impulsive semilinear differential equations.


The rest of the thesis consists a detailed account of the above topics.
Chapter - 2
In this chapter, the existence, uniqueness, stability via continuous dependence and Ulam stabilities of nonlinear integro-differential equations with random impulses are found out under sufficient condition. The results are obtained by using Leray-Schauder alternative fixed point theorem and Banach contraction principle.

2.1 Introduction

Mathematical modelling of real-life problems in many engineering and scientific disciplines usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations, stochastic equations. Many mathematical formulations of physical phenomena contain integro-differential equations. These equations arise in many fields like fluid dynamics, biological models and chemical kinetics. (For details, see [3,4,60,93] and the references therein.)

Impulsive differential equations are well known to model problems from many areas of science and engineering. There has been much research activity concerning the theory of impulsive differential equations see [46,67]. The impulses may exist at deterministic or random points.
There are a lot of papers which investigate the properties of deterministic impulses see [5,35,46,67] and the references therein. When the impulses exist at random points, the solution behaves as a stochastic process (see [6,8,84-88] and the references therein). In [7, 74,75,76] A.Vinodkumar and A. Anguraj studied the existence results for the random impulsive neutral functional differential equations and differential inclusions with delays. In [92], S.Zhang and J. Sun generalized the distribution of random impulses with the Erlang distribution.

The stabilities like continuous dependence, Hyers-Ulam stability, Hyers-Ulam-Rassias stability, exponential stability and asymptotic stability have attracted the attention of many mathematicians (see [14,36,49,66,73,78-82] and the references therein). In [78], J. Wang, L. Lv, and Y. Zhou have given the Ulam’s type stability and data dependence for fractional differential equations (FDEs). J. Wang, L. Lv, and Y. Zhou in [79] studied stability of FDEs using fixed point theorem in a generalized complete metric space. In [80], J. Wang, Y. Zhou, and M. Fečkan studied Ulam’s stability for the nonlinear impulsive FDEs. The technique developed in [46,67,86] is utilized.

This chapter is organized as follows: In section 2.2, the notations, definitions and preliminary facts which are used throughout this chapter are recalled briefly. In section 2.3, the existence of solutions of nonlinear
delay integro-differential equations with random impulses by using Leray-Schauder alternative fixed point theory is investigated. An interesting feature of this method is that this yields simultaneously the existence and maximal interval of existence and further investigated the existence and uniqueness of solutions of random impulsive nonlinear delay integro-differential equations by relaxing the linear growth condition.

In section 2.4, the stability through continuous dependence on initial conditions of random impulsive nonlinear delay integro-differential equations is discussed. The Hyers Ulam stability and Hyers Ulam-Rassias stability of the solutions of nonlinear delay integro-differential differential systems is investigated in section 2.5.

2.2 Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and $\Omega$ a nonempty set. Assume that $\tau_k$ is a random variable defined from $\Omega$ to $D_k = (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. Furthermore, assume that $\tau_k$ follow Erlang distribution, where $k = 1, 2, \ldots$ and let $\tau_i$ and $\tau_j$ are independent with each other as $i \neq j$ for $i, j = 1, 2, \ldots$. For the sake of simplicity, denote that $\mathbb{R}_r = [\tau_r, +\infty), \mathbb{R}^+ = [0, +\infty)$.

Consider the nonlinear delay integro-differential equation of the form:
where the functional \( F : \Delta \times \mathbb{C} \to \mathbb{R}^n \), \( \mathbb{C} = \mathbb{C}([-r,0], \mathbb{R}^n) \) is the set of piecewise continuous functions mapping \([-r,0]\) in to \( \mathbb{R}^n \) with some given \( r > 0 \); \( \sigma : \mathbb{R}^+ \to \mathbb{R}^+ \); \( \xi_0 = t_0 \) and \( \xi_k = \xi_{k-1} + \tau_k \) for \( k = 1, 2, \ldots \). Here \( t_0 \in \mathbb{R}_r \) is an arbitrary real number.

Obviously, \( t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \lim_{k \to \infty} \xi_k = \infty \); \( b_k : D_k \to R^{m \times n} \) is a matrix-valued function for each \( k = 1, 2, \ldots \); \( x(\xi_k) = \lim_{t \to \xi_k} x(t) \) according to their paths with the norm \( \| x \|_{P^t} = \sup_{t-r \leq s \leq t} | x(s) | \) for each \( t \) satisfying \( \tau \leq t \leq T \). \( \mathbb{P} \) is any given norm in \( X \), here \( \Delta \) denotes the set \( \{(t,s) : 0 \leq s \leq t < \infty \} \).

Denote \( \{ B_t, t \geq 0 \} \) the simple counting process generated by \( \{ \xi_n \} \), that is, \( \{ B_t \geq n \} = \{ \xi_n \leq t \} \), and denote \( \mathbb{F}_t \) the \( \sigma \)-algebra generated by \( \{ B_t, t \geq 0 \} \). Then \( (\Omega, \mathbb{P}, \{ \mathbb{F}_t \}) \) is a probability space. Let \( L_p = L_p(\Omega, \mathbb{F}_t, \mathbb{R}^n) \) denote the Banach space of all \( \mathbb{F}_t \)-measurable \( p^{th} \) integrable random variables with values in \( \mathbb{R}^n \).
Assume that $T > t_0$ is any fixed time to be determined later and let $\mathcal{B}$ denote the Banach space $\mathcal{B}([t_0-r,T],L_p)$, the family of all $\mathcal{F}_t$-measurable, $C$-valued random variables $\psi$ with the norm

$$p_{\mathcal{P}_\psi} = \left( \sup_{t_0 \leq t \leq T} E \mathcal{P}_\psi^p \right)^{1/p}.$$

Let $L^0_p(\Omega,\mathcal{B})$ denote the family of all $\mathcal{F}_0$-measurable, $\mathcal{B}$-valued random variable $\varphi$.

**Definition 2.2.1**

A map $F(t,s,x) : \Delta \times C \to X$, for all $t \in [\tau, T]$, $F(t,\cdot)$ satisfies $L^p$-Caratheodory, if

(i) $S \to F(t,s,x)$ is measurable for each $X \in C$;

(ii) $X \to F(t,s,x)$ is continuous for almost all $t \in [\tau, T]$;

(iii) for each positive integer $m > 0$, there exists $\alpha_m \in L^1([\tau, T],\mathbb{R}^+)$ such that $\sup_{\|x\| \leq m} E \| F(t,s,x) \|^p \leq \alpha_m(t)$, for $t \in [\tau, T]$.

**Definition 2.2.2**

For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B}, t_0 - r \leq t \leq T\}$ is called a solutions to equation (2.2.1) in $(\Omega, P, \{\mathcal{F}_t\})$, if
(i) \( x(t) \in \mathcal{N}^p \) is \( F_i \)-adapted for \( t \geq t_0 \);

(ii) \( x(t_0 + s) = \varphi(s) \in L_2^0(\Omega, F) \) when \( s \in [-r, 0] \)

\[
x(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{j=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \left[ \int_{\xi_{j-1}}^{\xi_j} [\int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu]ds \right] + \int_{t_0}^{t} \left[ \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu]ds \right] I_{[\xi_k, \xi_{k+1})}(t), t \in [t_0, T] \quad (2.2.2)
\]

where \( \prod_{j=m}^{n}(\cdot) = 1 \) as \( m > n \), \( \prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})\cdots b_i(\tau_i) \), and \( I_A(\cdot) \)

is the index function,

\[
i.e., \quad I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}
\]

The existence theorem is based on the following theorem, which is a version of the topological transversality theorem.

**Theorem 2.2.1**

Let \( B \) be a convex subset of a Banach space \( E \) and assume that \( 0 \in B \). Let \( F : B \to B \) be a completely continuous operator and let

\[
U(F) = \{ x \in B : x = \lambda Fx \text{ for some } 0 < \lambda < 1 \};
\]

then either \( U(F) \) is unbounded or \( F \) has a fixed point.

**2.3 Existence Results**

In this section, to prove the existence theorem by using the following hypothesis:
(H₁) : The function $F : [t₀, T] × [t₀, T] × ℂ → ℜⁿ$ is continuous, $F(t, s, 0) = 0$, and it satisfies the Lipschitz continuous with respect to $x$, i.e., $EF(t, s, x) ≤ L(t, s, XP₁P, XP₂P')EPX₁ - x₁P', (t, s) ∈ Δ, x₁, x₂ ∈ ℜⁿ$,

where $L : [t₀, T] × [t₀, T] × ℜ⁺ × ℜ⁺ → ℜ⁺$ and is monotonically nondecreasing with respect to the second and third arguments.

(H₂) : There exists a continuous function $p : [t₀, T] × [t₀, T] → (0, ∞)$ such that

$$EF(t, s, x)P ≤ p(t, s)H(EPxP'), (t, s) ∈ Δ, x ∈ ℜⁿ,$$

where $H : ℜ⁺ → (0, ∞)$ is a continuous nondecreasing function.

(H₃) : $σ : [t₀, T] → [t₀, T]$, is a continuous functions such that $σ(t) ≤ t$.

(H₄) : $E{\max_{j,k}\prod_{j=1}^{k}\|b_j(τ_j)\|}$ is uniformly bounded that there is $c > 0$ such that $E{\max_{j,k}\prod_{j=1}^{k}\|b_j(τ_j)\|} ≤ C$ for all $τ_j ∈ D_j, j = 1, 2, ⋯$.

**Theorem 2.3.1**

If the hypothesis (H₂)–(H₄) hold, then system (2.2.1) has a solution $X(t)$, defined on $[t₀, T]$ provided that the following inequality is satisfied.
\[ M_1 \int_0^T p(s,s)ds < \int_0^\infty \frac{ds}{H(s)}, \quad (2.3.1) \]

where \( M_1 = 2^{p-1} \max\{1,C^p\}(T-t_0)^2, \) \( c_1 = 2^{p-1}C^p E \Phi \Phi' \) and \( C^p \geq \frac{1}{2^{p-1}}. \)

**Proof**

Let \( T \) be an arbitrary number \( t_0 < T < +\infty \) satisfying (2.3.1), transforming the problem (2.2.1) into a fixed point problem. Consider the operator \( \Phi : B \rightarrow B \) defined by

\[
\Phi(x)(t) = \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{k} \prod_{j=i+1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[ \int_0^{s} F(s,\mu,x(\sigma(\mu)))d\mu \right] ds \right. \\
+ \left. \int_{\xi_k}^{s} F(s,\mu,x(\sigma(\mu)))d\mu \right] I_{[\xi_k,\xi_{k+1}]}(t), \quad t \in [t_0, T].
\]

In order to use the transversality theorem, first the prioriestimates are established for the solutions of the integral equation and \( \lambda \in (0,1), \)

\[
x(t) = \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{k} \prod_{j=i+1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[ \int_0^{s} F(s,\mu,x(\sigma(\mu)))d\mu \right] ds \right. \\
+ \left. \int_{\xi_k}^{s} F(s,\mu,x(\sigma(\mu)))d\mu \right] I_{[\xi_k,\xi_{k+1}]}(t), \quad t \in [t_0, T],
\]

Thus by \((H_2)-(H_4),\) it is found
\[ P_x(t) P^r \leq \lambda^r P \left[ \sum_{k=0}^{+\infty} P \prod_{i=1}^{k} b_i(\tau_i) \Phi(0) P \right. \\
+ \sum_{i=1}^{k} P \prod_{j=i}^{k} b_j(\tau_j) P \left[ \int_{\tau_{i-1}}^{\tau_i} F(s, \mu, x(\sigma(\mu))) d\mu \mathcal{P} ds \right] \\
+ \int_{\tau_k}^{t} P \left[ \int_{0}^{t} F(s, \mu, x(\sigma(\mu))) d\mu \mathcal{P} ds \right] I_{[\tau_k, \tau_{k+1})}(t) \right]^p \\
\leq 2^{p-1} \left\{ \sum_{k=0}^{+\infty} P \prod_{i=1}^{k} b_i(\tau_i) P \left[ \Phi(0) P \right] \mathcal{P} I_{[\tau_k, \tau_{k+1})}(t) \right\} \\
\left\{ \sum_{k=0}^{+\infty} \sum_{i=1}^{k} P \prod_{j=i}^{k} b_j(\tau_j) P \left[ \int_{\tau_{i-1}}^{\tau_i} F(s, \mu, x(\sigma(\mu))) d\mu \mathcal{P} ds \right] \\
+ \int_{\tau_k}^{t} P \left[ \int_{0}^{t} F(s, \mu, x(\sigma(\mu))) d\mu \mathcal{P} ds \right] I_{[\tau_k, \tau_{k+1})}(t) \right\}^p \right]\] \\
\leq 2^{p-1} \max \left\{ \prod_{i=1}^{k} b_i(\tau_i) P \left[ \Phi(0) P \right] \mathcal{P} \right\} \\
+ 2^{p-1} \left[ \max \left\{ 1, \prod_{j=i}^{k} \left\| b_j(\tau_j) \right\| \right\} \right]^p \left( \int_{0}^{t} \left\| F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds \right)^p \right]\] \\
Noting that the last term of the right hand side of the above inequality increases in \( t \) and choose \( C^p \geq \frac{1}{2^{p-1}} \), the following is obtained \\
\[ P_x P^r \leq 2^{p-1} \max \left\{ \prod_{i=1}^{k} b_i(\tau_i) P \left[ \Phi P \right] \mathcal{P} \right\} \\
+ 2^{p-1} \left[ \max \left\{ 1, \prod_{j=i}^{k} \left\| b_j(\tau_j) \right\| \right\} \right]^p (T - t_0) \int_{0}^{t} \left\| F(s, \mu, x(\sigma(\mu))) d\mu \right\| ds \]}
then

\[ \begin{align*}
E \mathbb{P}_x \mathbb{P}_v & \leq 2^{p-1} C^p E \left[ \mathbb{P}_x \mathbb{P}_v^p \right] + 2^{p-1} \max \{1, C^p\} (T - t_0) \left( \int_0^{s_E} E \|F(s, \mu, x(\mu))d\mu\|^p \right) ds \\
& \leq 2^{p-1} C^p E \left[ \|\phi\|^p \right] + 2^{p-1} \max \{1, C^p\} (T - t_0) \left( \int_0^{s_E} p(s, \mu) H \left( E \mathbb{P}_x(\sigma(\mu)) \mathbb{P}_y \right) d\mu \right) ds \\
& \leq 2^{p-1} C^p E \left[ \|\phi\|^p \right] + 2^{p-1} \max \{1, C^p\} (T - t_0)^2 \left( \int_0^{s_E} p(s, s) H \left( E \mathbb{P}_x(\sigma(\mu)) \mathbb{P}_y \right) d\mu \right) ds
\end{align*} \]

Because the last term of the right hand side of the above inequality also increases in \( t \), then

\[ \sup_{t_0 \leq s \leq t} E \mathbb{P}_x \mathbb{P}_v \]

\[ \leq 2^{p-1} C^p E \left[ \|\phi\|^p \right] + 2^{p-1} \max \{1, C^p\} (T - t_0)^2 \left( \int_0^{s_E} p(s, s) H \left( \sup_{t_0 \leq s \leq s_E} E \mathbb{P}_x \mathbb{P}_v \right) d\mu \right) ds. \]

Consider the function \( \ell(t) \) defined by

\[ \ell(t) = \sup_{t_0 \leq s \leq t} E \mathbb{P}_x \mathbb{P}_v, \quad t \in [t_0, T]. \]

Then, for any \( t \in [t_0, T] \) it follows that

\[ \ell(t) \leq 2^{p-1} C^p E \left[ \|\phi\|^p \right] + 2^{p-1} \max \{1, C^p\} (T - t_0)^2 \left( \int_0^{s_E} p(s, s) H(\ell(s)) ds \right). \quad (2.3.2) \]
Denoting by \( u(t) \) the right hand side of the above inequality (2.3.2), then

\[
\ell(t) \leq u(t), \quad t \in [t_0, T],
\]

\[
u(t_0) = 2^{p-1} C^p \| \varphi \|^p = c_i
\]

and

\[
u'(t) = 2^{p-1} \max\{1, C^p\} (T - t_0)^2 p(t,t) H(\ell(t))
\]

\[
\leq 2^{p-1} \max\{1, C^p\} (T - t_0)^2 p(t,t) H(u(t)), \quad t \in [t_0, T].
\]

Then

\[
\frac{\nu'(t)}{H(u(t))} \leq 2^{p-1} \max\{1, C^p\} (T - t_0)^2 p(t,t), \quad t \in [t_0, T]. \quad (2.3.3)
\]

Integrating (2.3.3) from \( t_0 \) to \( t \) and by making use of the change of variable, now

\[
\int_{\nu(t_0)}^{\nu(t)} \frac{ds}{H(s)} \leq 2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_0^t p(s,s)ds
\]

\[
\leq 2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_0^T p(s,s)ds
\]

\[
< \int_{\nu(t_0)}^{\nu(t)} \frac{ds}{H(s)}, \quad t \in [t_0, T], \quad (2.3.4)
\]

where the last inequality is obtained by (2.3.1). From (2.3.4) and by mean value theorem, there is a constant \( \eta_1 \) such that \( u(t) \leq \eta_1 \) and hence

\[
\ell(t) \leq \eta_1. \quad \text{Since} \quad \sup_{t_0 \leq \tau \leq t} E_P \mathbb{P}_\tau^P = \ell(t) \quad \text{holds for every} \quad t \in [t_0, T], \quad \text{then}
\]

45
sup \( E \mathbb{P}_x \mathbb{P}_y \leq \eta \), where \( \eta \) only depends on \( T \), the functions \( p \) and \( H \), and consequently

\[
E \mathbb{P}_x \mathbb{P}_y = \sup_{t_0 \leq s \leq T} E \mathbb{P}_x \mathbb{P}_y \leq \eta.
\]

In the next steps, \( \Phi \) is continuous and completely continuous is proved.

**Step 1**

**To Prove that \( \Phi \) is Continuous**

Let \( \{x_n\} \) be a convergent sequence of elements of \( x \) in \( B \). Then

for each \( t \in [t_0, T] \),

\[
\Phi_{x_n}(t) = \sum_{k=0}^{\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \right] \left( \int_{\xi_{i-1}}^{\xi_i} F(s, \mu, x_n[\sigma(\mu)])d\mu ds \right)
\]

\[
+ \sum_{i=1}^{\infty} \left[ \int_{\xi_{i-1}}^{\xi_i} F(s, \mu, x_n[\sigma(\mu)])d\mu ds \right] I_{(\xi_k, \xi_{k+1})}(t).
\]

Thus,

\[
\Phi_{x_n}(t) - \Phi_x(t) = \sum_{k=0}^{\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) \right] \left( \int_{\xi_{i-1}}^{\xi_i} F(s, \mu, x_n[\sigma(\mu)])d\mu - \int_{0}^{s} F(s, \mu, x[\sigma(\mu)])d\mu \right) ds
\]

\[
+ \sum_{i=1}^{\infty} \left[ \int_{\xi_{i-1}}^{\xi_i} F(s, \mu, x_n[\sigma(\mu)])d\mu - \int_{0}^{s} F(s, \mu, x[\sigma(\mu)])d\mu \right] ds I_{(\xi_k, \xi_{k+1})}(t).
\]

and

\[
E \mathbb{P}_x \Phi_{x_n} - \Phi_x \mathbb{P}_y \leq \max \{1, C^n\}(T - t_0)
\]
Thus $\Phi$ is clearly continuous.

**Step 2**

**To Prove that $\Phi$ is Completely Continuous Operator**

Denote $B_m = \{x \in B \mid P_x \mathbb{P} \leq m\}$ for some $m \geq 0$.

**Step 2.1**

**To Show that $\Phi$ Maps $B_m$ into an Equicontinuous Family**

Let $y \in B_m$ and $t_1, t_2 \in [t_0, T]$. If $t_0 < t_1 < t_2 < T$, then by using hypotheses $(H_2)-(H_4)$ and condition (2.3.1), the following is obtained,

\[
\Phi x(t_1) - \Phi x(t_2) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} F(s, \mu, x(\sigma(\mu))) d\mu ds
\]

\[
+ \int_{\xi_k}^{t_1} \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds \] \[I_{[\xi_k, \xi_{k+1}]}(t_1)
\]

\[
- \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} F(s, \mu, x(\sigma(\mu))) d\mu ds
\]

\[
+ \int_{\xi_k}^{t_2} \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds \] \[I_{[\xi_k, \xi_{k+1}]}(t_2).
\]
Thus,

$$\Phi_x(t_1) - \Phi_x(t_2)$$

$$= \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{+\infty} \prod_{j=i}^{k} b_j(\tau_j) \int_{s_{i-1}}^{s_i} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

$$+ \int_{s_{k+1}}^{s_k} F(s, \mu, x(\sigma(\mu))) d\mu ds \left( I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2) \right)$$

$$+ \sum_{i=1}^{+\infty} \prod_{j=i}^{k} b_j(\tau_j) \int_{s_{i-1}}^{s_i} F(s, \mu, x(\sigma(\mu))) d\mu ds \left( I_{[\xi_k, \xi_{k+1}]}(t_2) \right).$$

Then,

$$E x^1 \Phi_x(t_1) - \Phi_x(t_2) x^1 \leq 2^{p-1} E x^1 P^1 + 2^{p-1} E x^2 P^2,$$  

$$= \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{+\infty} \prod_{j=i}^{k} b_j(\tau_j) \int_{s_{i-1}}^{s_i} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

$$+ \int_{s_{k+1}}^{s_k} F(s, \mu, x(\sigma(\mu))) d\mu ds \left( I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2) \right)$$

and

$$I_2 = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) \int_{s_{i-1}}^{s_i} F(s, \mu, x(\sigma(\mu))) d\mu ds \left( I_{[\xi_k, \xi_{k+1}]}(t_2) \right).$$

Furthermore,

$$E x^1 P^1 \leq 2^{p-1} C \varphi(0) P^1 E (I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2))$$

$$+ 2^{p-1} \max \left\{ 1, C \varphi(0) \right\} (t_1 - t_0) E \int_{0}^{t_1} P^1 \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

$$+ 2^{p-1} \max \left\{ 1, C \varphi(0) \right\} (t_1 - t_0) E \int_{0}^{t_1} P^1 \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

$$+ 2^{p-1} \max \left\{ 1, C \varphi(0) \right\} (t_1 - t_0) E \int_{0}^{t_1} P^1 \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

$$+ 2^{p-1} \max \left\{ 1, C \varphi(0) \right\} (t_1 - t_0) E \int_{0}^{t_1} P^1 \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

$$+ 2^{p-1} \max \left\{ 1, C \varphi(0) \right\} (t_1 - t_0) E \int_{0}^{t_1} P^1 \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

$$+ 2^{p-1} \max \left\{ 1, C \varphi(0) \right\} (t_1 - t_0) E \int_{0}^{t_1} P^1 \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds$$

48
\[
\times E(I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2)) \\
\leq 2^{p-1} C^p E \mathbb{P} \varphi(0) \mathbb{P}' E(I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2)) \\
+ 2^{p-1} \max \left\{ 1, C^p \right\} (t_1 - t_0)^2 \int_0^t \int_s^t p(s, \mu) H(E \mathbb{P} x \mathbb{P}') d \mu ds \\
\leq 2^{p-1} C^p E \mathbb{P} \varphi(0) \mathbb{P}' E(I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2)) \\
+ 2^{p-1} \max \left\{ 1, C^p \right\} (t_1 - t_0)^2 \int_0^t p(s, s) H(E \mathbb{P} x \mathbb{P}') ds \\
\times E(I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2)) \\
\leq 2^{p-1} C^p E \mathbb{P} \varphi(0) \mathbb{P}' E(I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2)) \\
+ 2^{p-1} \max \left\{ 1, C^p \right\} (t_1 - t_0)^2 E \int_0^t M^* H(E(m)) ds \\
\times E(I_{[\xi_k, \xi_{k+1}]}(t_1) - I_{[\xi_k, \xi_{k+1}]}(t_2)) \\
\rightarrow 0 \quad \text{as} \quad t_2 \rightarrow t_1, \quad (2.3.6)
\]

where \( M^* = \sup\{p(t, t) : t \in [t_0, T]\} \), and

\[
E \mathbb{P} I_{2} \mathbb{P}' \leq C^p (t_2 - t_1) E \int_{t_1}^{t_2} \mathbb{P}' \int_0^s F(s, \mu, \mathbb{x}(\sigma(\mu))) d \mu \mathbb{P}' ds \\
\leq C^p (t_2 - t_1)^2 \int_{t_1}^{t_2} M^* H(m) ds \quad (2.3.7)
\]

\rightarrow 0 \quad \text{as} \quad t_2 \rightarrow t_1.
The right hand side of (2.3.6) and (2.3.7) is independent of \( x \in B_m \).

It follows that the right hand side of (2.3.5) tends to zero as \( t_2 \to t_1 \). Thus, \( \Phi \) maps \( B_m \) into an equicontinuous family of functions.

**Step 2.2**

**To Show that \( \Phi B_m \) is Uniformly Bounded**

From (2.3.1), \( P_x P_B \leq m \) and by \((H_3)-(H_5)\) it yields that

\[
P(\Phi x)(t) P^p \leq 2^{p-1} \max\{\prod_{k=1}^{k} P b_i(\tau_j) P^p \} P \varphi(0) P^p
\]

\[
+ 2^{p-1} \left[ \max\{1,\prod_{k=1}^{k} P b_i(\tau_j) P^p \} \sum_{s=0}^{\infty} \left[ \sum_{k=0}^{\infty} 0_t F(s, \mu, x(\sigma(\mu))) d\mu P ds I_{(\xi_k, \xi_{k+1})}(t) \right]^p \right].
\]

Thus,

\[
E P(\Phi x) P^p \leq 2^{p-1} C^p E P \varphi(0) P^p
\]

\[
+ 2^{p-1} \max\{1, C^p\} (T - t_0) \int_0^T E \left[ \sum_{s=0}^{\infty} 0_t F(s, \mu, x(\sigma(\mu))) d\mu P^p ds \right.
\]

\[
E P(\Phi x) P^p \leq 2^{p-1} C^p E P \varphi(0) P^p + 2^{p-1} \max\{1, C^p\} (T - t_0)^2 P \alpha_m P^l.
\]

This yields that the set \( \{(\Phi x)(t), \, P_x P_B \leq m\} \) is uniformly bounded, so \( \Phi B_m \) is uniformly bounded. Already shown that \( \Phi B_m \) is equicontinuous collection. Now it is sufficient, by the Arzela - Ascoli theorem, to show that \( \Phi \) maps \( B_m \) into a precompact set in \( \mathbb{R}^n \).
Step 2.3

To Show that $\Phi B_m$ is Compact.

Let $t_0 < t \leq T$ be fixed and $\varepsilon$ a real number satisfying $\varepsilon \in (0, t - t_0)$, for $x \in B_m$. Now define

\[
(\Phi_\varepsilon x)(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(t) \varphi(0) + \sum_{j=1}^{k} \prod_{i=1}^{k} b_j(t) \int_{0}^{\varepsilon} \int_{r-i-\varepsilon}^{r-i} F(s, \mu, x(\sigma(\mu)))d\mu ds
\]

\[
+ \int_{\varepsilon_i}^{t-i-\varepsilon} \int_{r-i-\varepsilon}^{r-i} F(s, \mu, x(\sigma(\mu)))d\mu ds]t_{i-1}, t \in (t_0, t - \varepsilon). \tag{2.3.8}
\]

The set $H_\varepsilon (t) = \{(\Phi_\varepsilon x)(t) : x \in B_m\}$ is precompact in $\mathbb{R}^n$ for every $\varepsilon \in (0, t - t_0)$. By using $(H_2) - (H_4)$, (2.3.1) and $E P x P_t^\varepsilon \leq m$,

\[
E P(\Phi x) - (\Phi_\varepsilon x) P_t^\varepsilon \leq \max \{1, C^p\} (T - t_0) \int_{t-\varepsilon}^{t} M^s H(m)ds.
\]

Therefore, there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_m\}$. Hence the set $\{(\Phi x)(t) : x \in B_m\}$ is precompact in $\mathbb{R}^n$. Therefore, $\Phi$ is a completely continuous operator.

Moreover, the set $U(\Phi) = \{x \in \mathcal{B} : x = \lambda \Phi x, \text{ for some } 0 < \lambda < 1\}$ is bounded. Consequently, by Theorem 2.2.1, the operator $\Phi$ has a fixed point in $\mathcal{B}$. Therefore, the system (2.2.1) has a solution. Thus, the proof is completed.
Now, another existence result for the system (2.2.1) is given by means of Banach contraction principle.

**Theorem 2.3.2**

If the hypothesis \((H_1),(H_3)\) and \((H_4)\) holds then the initial value problem (2.2.1) has a unique solution on \([t_0,T]\).

**Proof**

Consider the nonlinear operator \(\Phi: \mathcal{B} \rightarrow \mathcal{B}\) defined as in Theorem 2.3.1

\[
E\|\Phi x - \Phi y\|_\mathcal{B}^p \leq 2^{p-1} \max\{1,C^p\}(T-t_0)\int_0^t\left\{\int_0^s EPF(s,\mu,x(\sigma(\mu)))d\mu - \int_0^s F(s,\mu,y(\sigma(\mu)))d\mu\right\}ds
\]

\[
\leq 2^{p-1} \max\{1,C^p\}(T-t_0)\int_0^t\left\{\int_0^s L(s,\mu,E P\! x(\sigma(\mu)))\mathcal{P}'\right\} ds
\]

\[
\leq 2^{p-1} \max\{1,C^p\}(T-t_0)\int_0^t\left\{\int_0^s L(s,\mu,E P\! x \mathcal{P}',E P\! y)\right\} ds
\]

\[
\leq 2^{p-1} \max\{1,C^p\}(T-t_0)^2\int_0^t L(s,s,E P\! x \mathcal{P}',E P\! y) ds.
\]

Taking supremum over \(t\), the result is

\[
\|\Phi x - \Phi y\|_\mathcal{B}^p \leq \Lambda(T)P_x - y P'_y,
\]

with \(\Lambda(T) = 2^{p-1} \max\{1,C^p\}(T-t_0)^2\int_0^t L(s,s,E P\! x \mathcal{P}',E P\! y) ds\).
Take a suitable $0 < T_1 < T$ sufficient small such that $\Lambda(T_1) < 1$, and hence $\Phi$ is a contraction on $B_{r_1}$ ( $B_{r_1}$ denotes $B$ with $T$ substituted by $T_1$). Thus, by the well-known Banach fixed point theorem a unique fixed point $x \in B_{r_1}$ is obtained for operator $\Phi$, and hence $\Phi x = x$ is a solution of the system (2.2.1). This procedure can be repeated to extend the solution to the entire interval $[-r,T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of solutions on the whole interval $[-r,T]$.

2.4 Continuous Dependence

In this section, the stability of the system (2.2.1) through the continuous dependence of solutions on initial condition is analyzed.

Theorem 2.4.1

Let $x(t)$ and $\bar{x}(t)$ be solutions of the system (2.2.1) with initial values $\varphi(0)$ and $\bar{\varphi}(0) \in B$ respectively. If the assumptions of Theorem 2.3.2 is satisfied, then the solution of the system (2.2.1) is stable in the mean square.

Proof

By the assumption, $x$ and $\bar{x}$ are the two solutions of the system (2.2.1) for $t \in [t_0, T]$. Then,
\[ x(t) - \bar{x}(t) = \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i)[\varphi(0) - \varphi(0)] \right] \\
+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[ \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu - \int_{0}^{s} F(s, \mu, \bar{x}(\sigma(\mu)))d\mu \right]ds \\
+ \left[ \int_{\xi_k}^{s} F(s, \mu, x(\sigma(\mu)))d\mu - \int_{0}^{s} F(s, \mu, \bar{x}(\sigma(\mu)))d\mu \right]ds \right] I_{[\xi_k, \xi_{k+1}]}(t). \]

By using the hypotheses \((H_1), (H_3)\) and \((H_4)\), the following is found out

\[ E P x - x P' \]

\[ \leq 2^{p-1} \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_i(\tau_i) P' E P \varphi(0) - \varphi(0) P' I_{[\xi_k, \xi_{k+1}]}(t) \]

\[ + 2^{p-1} E \left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_j(\tau_j) P x \int_{\xi_{i-1}}^{\xi_i} \left[ \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu \right]ds \right] \]

\[ - \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu P ds + \int_{\xi_k}^{s} P \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu \]

\[ - \left[ \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu P ds \right] I_{[\xi_k, \xi_{k+1}]}(t) \]

\[ \leq 2^{p-1} E \left\{ \max_{k} \left( \prod_{i=1}^{k} \left[ P b_i(\tau_i) P' \right] \right) \right\} E P \varphi(0) - \varphi(0) P' \]

\[ + 2^{p-1} E \left[ \max_{k} \left( \prod_{i=1}^{k} \left[ b_j(\tau_j) \right] \right) \right] \times E \left( \int_{0}^{s} P \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu \right) \]

\[ - \int_{0}^{s} F(s, \mu, x(\sigma(\mu)))d\mu P ds I_{[\xi_k, \xi_{k+1}]}(t) \]

54
By applying Grownwall’s inequality,

\[ \leq 2^{p-1} C^p E \varphi(0) - \varphi(0) \mathbb{P} \]

\[ + 2^{p-1} \max\{1, C^p\} (t - t_0) \int_0^t E P \int_0^t F(s, \mu, x(\sigma(\mu))) \, d\mu \]

\[ - \int_0^t F(s, \mu, \bar{x}(\sigma(\mu))) \, d\mu \mathbb{P} \, ds \]

\[ \sup_{t \in [t_0, T]} E \mathbb{P} x - \bar{x}_t \mathbb{P} \leq 2^{p-1} C^p E \varphi(0) - \varphi(0) \mathbb{P} \]

\[ + 2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_0^t L(s, \bar{s}, E \mathbb{P} x, E \mathbb{P} y) \sup_{t \in [t_0, t]} E \mathbb{P} x - \bar{x}_t \mathbb{P} ds. \]

By applying Grownwall’s inequality,

\[ \sup_{t \in [t_0, T]} E \mathbb{P} x - \bar{x}_t \mathbb{P} \leq 2^{p-1} C^p E \varphi(0) - \varphi(0) \mathbb{P} \]

\[ \times \exp(2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_0^t L(s, \bar{s}, E \mathbb{P} x, E \mathbb{P} y) \, ds) \]

\[ \leq \Gamma E \varphi(0) - \varphi(0) \mathbb{P}, \]

where,

\[ \Gamma = 2^{p-1} C^p \exp(2^{p-1} \max\{1, C^p\} (T - t_0)^2 \int_0^t L(s, \bar{s}, E \mathbb{P} x, E \mathbb{P} y) \, ds) \]

Now given \( \varepsilon > 0 \), choose \( \delta = \frac{\varepsilon}{\Gamma} \) such that \( E \mathbb{P} \varphi(0) - \varphi(0) \mathbb{P} < \delta \).

Then \( \sup_{t \in [t_0, T]} E \mathbb{P} x - \bar{x}_t \mathbb{P} \leq \varepsilon. \)

This completes the proof.
2.5 Ulam -Hyers- Rassias Type Stability

In this section, the Ulam- Hyers stability of random impulsive semilinear differential equations (2.2.1) is found out.

Let $\varepsilon > 0$, $\mu > 0$ and $\phi:[t_o,T] \rightarrow \mathbb{R}^+$ be a piecewise continuous function. Consider the following inequalities:

\[
\begin{align*}
E P x'(t) - \int_0^t F(t,s,x(\sigma(s)))ds P^p & \leq \varepsilon, \quad t \neq \xi_k, \quad t \geq t_0. \\
E P \{x(\xi_k) - b_k(\tau_k)x(\xi^-_k)\} P^p & \leq \varepsilon, \quad k = 1,2,.....
\end{align*}
\] (2.5.1)

\[
\begin{align*}
E P x'(t) - \int_0^t F(t,s,x(\sigma(s)))ds P^p & \leq \phi(t), \quad t \neq \xi_k, \quad t \geq t_0. \\
E P \{x(\xi_k) - b_k(\tau_k)x(\xi^-_k)\} P^p & \leq \mu, \quad k = 1,2,.....
\end{align*}
\] (2.5.2)

\[
\begin{align*}
E P x'(t) - \int_0^t F(t,s,x(\sigma(s)))ds P^p & \leq \varepsilon\phi(t), \quad t \neq \xi_k, \quad t \geq t_0. \\
E P \{x(\xi_k) - b_k(\tau_k)x(\xi^-_k)\} P^p & \leq \varepsilon\mu, \quad k = 1,2,.....
\end{align*}
\] (2.5.3)

**Definition 2.5.1**

The system (2.2.1) is Ulam- Hyers stable in the $p^{th}$ mean if there exists a real number $\kappa > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathbb{B}$ of the inequality (2.5.1) there exists a solution $y \in \mathbb{B}$ of the system (2.2.1) with $E P x(t) - y(t) P^p \leq \kappa \varepsilon$, $t \in [t_0,T]$.

**Definition 2.5.2**

The system (2.2.1) is generalized Ulam- Hyers stable in the $p^{th}$ mean if there exists a real number $\eta \in \mathbb{B}, \eta(0) = 0$ such that for each
solution \( x \in B \) of the inequality (2.5.1) there exists a solution \( y \in B \) of the system (2.2.1) with \( \mathbb{E} \mathcal{P}_x(t) - y(t) \mathcal{P} \leq \eta(\varepsilon), \quad t \in [t_0, T]. \)

**Definition 2.5.3**

The system (2.2.1) is Ulam-Hyers-Rassias stable in the \( p^{th} \) mean with respect to \((\phi, \mu)\) if there exists a real number \( \zeta > 0 \) such that for each \( \varepsilon > 0 \) and for each solution \( x \in B \) of the inequality (2.5.3) there exists a solution \( y \in B \) of the system (2.2.1) with

\[
\mathbb{E} \mathcal{P}_x(t) - y(t) \mathcal{P} \leq \zeta \varepsilon (\phi(t) + \mu), \quad t \in [t_0, T].
\]

**Definition 2.5.4**

The system (2.2.1) is generalized Ulam-Hyers-Rassias stable in the \( p^{th} \) mean with respect to \((\phi, \mu)\) if there exists a real number \( \zeta > 0 \) such that for each solution \( x \in B \) of the inequality (2.5.2) there exists a mild solution \( y \in B \) of the system (2.2.1) with

\[
\mathbb{E} \mathcal{P}_x(t) - y(t) \mathcal{P} \leq \zeta (\phi(t) + \mu), \quad t \in [t_0, T].
\]

**Remark 2.5.1**

It is clear that

1. Definition (2.5.1) \( \Rightarrow \) Definition (2.5.2)
2. Definition (2.5.3) \( \Rightarrow \) Definition (2.5.4)
3. Definition (2.5.3) for \( \phi(t) = \mu = 1 \) \( \Rightarrow \) Definition (2.5.1).
**Remark 2.5.2**

A function $x \in B$ is a solution of the inequality (2.5.3) if and only if there exists a function $h \in B$ and the sequence $h_k, k = 1,2,\ldots$ (which depend on $x$) such that

(i): $E \Phi(h(t)) \leq \varepsilon \phi(t), t \in [t_0, T]$ and $E \Phi h_k \leq \varepsilon \mu, k = 1,2,\ldots$;

(ii): $x'(t) = \int_0^t F(t,s,x(\sigma(s)))ds + h(t), t \neq \xi_k, t \geq t_0$;

(iii): $x(\xi_k) = b_k(\tau_k)x(\xi_k^-) + h_k, k = 1,2,\ldots$

One can have similar remarks for the inequalities (2.5.1) and (2.5.2).

**Remark 2.5.3**

If $x \in B$ is a solution of the inequality (2.5.3) then $x$ is a solution of the following integral inequality

$$
E \left\| x(t) - \sum_{k=0}^{\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) \phi(0) + \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \left[ \int_0^\xi F(s,\mu,x(\sigma(\mu)))d\mu \right] ds \right] \int_{[\xi_k,\xi_{k+1})}(t) \right\|^p
$$

$$
\leq 2^{n-1} \varepsilon \{ C^\alpha \mu + \max \{ 1, C^\alpha \} (T-t_0) \int_0^t \phi(s) ds \}, t \in [t_0, T].
$$

From the Remark 2.5.2,

$$
\begin{align*}
x'(t) &= \int_0^t F(t,s,x(\sigma(s)))ds + h(t), t \neq \xi_k, t \geq t_0. \\
x(\xi_k) &= b_k(\tau_k)x(\xi_k^-) + h_k, k = 1,2,\ldots
\end{align*}
$$

(2.5.4)
Then

\[ x(t) = \sum_{k=0}^{\infty} \prod_{j=1}^{k} b_j(\tau_j) \varphi(0) + \prod_{j=1}^{k} b_j(\tau_j) h_j \]

\[ + \sum_{i=0}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds + \int_{\xi_{k+1}}^{t} \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds \]

\[ + \sum_{i=0}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} h(s) ds + \int_{\xi_{k+1}}^{t} h(s) ds] I_{[\xi_k, \xi_{k+1}]} (t), \quad t \in [0, T]. \]

Therefore,

\[ E P x(t) - \sum_{k=0}^{\infty} \prod_{j=1}^{k} b_j(\tau_j) \varphi(0) + \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds \]

\[ + \int_{\xi_{k+1}}^{t} \int_{0}^{s} F(s, \mu, x(\sigma(\mu))) d\mu ds I_{[\xi_k, \xi_{k+1}]} (t) P^P \]

\[ = E P \sum_{k=0}^{\infty} \prod_{j=1}^{k} b_j(\tau_j) h_j + \sum_{i=0}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} h(s) ds + \int_{\xi_{k+1}}^{t} h(s) ds I_{[\xi_k, \xi_{k+1}]} (t) P^P \]

\[ \leq 2^{p-1} E \left\{ \max_{i=1}^{k} \prod_{j=1}^{k} \overline{P} b_j(\tau_j) P^P \right\} E P h_i P^P \]

\[ + 2^{p-1} E \left[ \max_{i=1}^{p} \prod_{j=1}^{k} \overline{P} b_j(\tau_j) P \right]^p (T-t_0) \int_{0}^{t} E P h(s) P^P ds \]

\[ \leq 2^{p-1} \varepsilon \{ C^P \mu + \max_{i=1}^{p} \{ 1, C^P \} (T-t_0) \int_{0}^{t} \phi(s) ds \}. \]

There are similar remarks for the solutions of the inequalities (2.5.1) and (2.5.2).

Now, Ulam-Hyers-Rassias stability results are given, in this section.
Theorem 2.5.1

Assumption $(H_1)$, $(H_3)$ and $(H_4)$ hold. Suppose there exists $\lambda > 0$ such that

$$\int_{t_0}^{t'} \phi(s) ds \leq \lambda \phi(t), \text{ for each } t \in [t_0, T],$$

where $\phi : [t_0, T] \to \mathbb{R}^+$ is a continuous nondecreasing function. Then the system (2.2.1) is Ulam-Hyers-Rassias stable in the $p^{th}$ mean.

Proof

Let $x \in B$ be a solution of the inequality (2.5.3). By Theorem 2.3.2 there exist a unique solution $y$ of the random impulsive delay integro-differential system

$$\begin{cases}
y'(t) = \int_{0}^{t} F(t, s, y(\sigma(s))) ds, \quad t \neq \xi_k, \quad t \geq t_0 \\
y(\xi_k) = b_k(\tau_k)y(\xi_k^-), \quad k = 1, 2, \ldots \quad (2.5.5) \\
y_0 = \varphi.
\end{cases}$$

Then

$$y(t + t_0) = \varphi(t), \quad \text{for } t \in [-r, 0],$$

$$y(t) = \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{j=1}^{k} b_j(\tau_j) \int_{\xi_j}^{\xi_j^+} \left[ \int_{0}^{s} F(s, \mu, y(\sigma(\mu))) d\mu \right] ds \right]$$

$$+ \sum_{k=0}^{+\infty} \left[ \int_{\xi_k}^{\xi_k^+} F(s, \mu, y(\sigma(\mu))) d\mu \right] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T].$$
By differential inequality (2.5.3),

\[
E P x(t) - \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0) + \sum_{i=1}^{k} \prod_{j=1}^{i} b_j(\tau_j) \left[ \int_{\tau_{i-1}}^{\tau_i} F(s, \mu, x(\sigma(\mu))) d\mu \right] ds
\]

\[
+ \int_{\tau_k}^{\tau_{k+1}} \left[ \int_{\mu}^{\xi_k} F(s, \mu, x(\sigma(\mu))) d\mu \right] ds I_{(\xi_k, \xi_{k+1})}(t) P' \]

\[
= E P \sum_{k=0}^{\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) h_i \right] + \sum_{i=1}^{k} \prod_{j=1}^{i} b_j(\tau_j) \left[ \int_{\tau_{i-1}}^{\tau_i} h(s) ds + \int_{\tau_k}^{\tau_{k+1}} h(s) ds \right] I_{(\xi_k, \xi_{k+1})}(t) P' \]

\[
\leq 2^{p-1} E \left\{ \max \left\{ \prod_{i=1}^{k} P h_i(\tau_i) P' \right\} \right\} E P h_i P' \]

\[
+ 2^{p-1} E [\max \{1, \prod_{i=1}^{k} P h_i(\tau_i) P' \}] (T - t_0) \times \varepsilon \int_{0}^{\tau} \phi(s) ds \]

\[
\leq 2^{p-1} \varepsilon [C^p \mu + \max \{1, C^p \} (T - t_0) \lambda \phi(t)] , \ t \in [t_0, T].
\]

Hence for each \( t \in [t_0, T] \),

\[
E P x(t) - y(t) P' = E P x(t) - \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0)
\]

\[
+ \sum_{i=1}^{k} \prod_{j=1}^{i} b_j(\tau_j) \left[ \int_{\tau_{i-1}}^{\tau_i} F(s, \mu, y(\sigma(\mu))) d\mu \right] ds
\]

\[
+ \int_{\tau_k}^{\tau_{k+1}} \left[ \int_{\mu}^{\xi_k} F(s, \mu, y(\sigma(\mu))) d\mu \right] ds I_{(\xi_k, \xi_{k+1})}(t) P'
\]

\[
E P x - y P' \leq 2^{p-1} E P x(t) - \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i) \varphi(0)
\]

\[
+ \sum_{i=1}^{k} \prod_{j=1}^{i} b_j(\tau_j) \left[ \int_{\tau_{i-1}}^{\tau_i} F(s, \mu, x(\sigma(\mu))) d\mu \right] ds
\]
\[ + \int_{\tau_k}^t \left[ \int_0^s F(s, \mu, x(\sigma(\mu)))d\mu ds \right] I_{[\tau_k, \tau_{k+1}]}(t) \, \mathbb{P} \]

\[ + \frac{2^{p-1} \mathbb{E} \sum_{i=1}^k \prod_{j=1}^k b(\tau_j) \int_{\tau_{i-1}}^{\tau_i} \left[ \int_0^s F(s, \mu, x(\sigma(\mu)))d\mu \right] ds}{\int_0^T \left[ \int_0^s F(s, \mu, y(\sigma(\mu)))d\mu \right] ds} \mathbb{P} \]

\[ - \int_{\tau_k}^t \left[ \int_0^s F(s, \mu, y(\sigma(\mu)))d\mu ds \right] I_{[\tau_k, \tau_{k+1}]}(t) \, \mathbb{P} \]

\[ \leq 2^{p-1} \varepsilon \{ C' \mu + \max \{ 1, C'' \} (T - t_0) \lambda \phi(t) \} \]

\[ + 2^{p-1} \max \{ 1, C'' \} (T - t_0)^2 \int_{t_0}^T L(s, s, E \mathbb{P} x \mathbb{P}', E \mathbb{P} y \mathbb{P}') \mathbb{E} \mathbb{P} x - y \mathbb{P}_s ds. \]

\[ \sup_{t \in [t_0, T]} \mathbb{E} \mathbb{P} x - y \mathbb{P}_t \leq 2^{p-1} \varepsilon \{ C' \mu + \max \{ 1, C'' \} (T - t_0) \lambda \phi(t) \} \]

\[ + 2^{p-1} \max \{ 1, C'' \} (T - t_0)^2 \int_{t_0}^T L(s, s, E \mathbb{P} x \mathbb{P}', E \mathbb{P} y \mathbb{P}') \sup_{s \in [t_0, t]} \mathbb{E} \mathbb{P} x - y \mathbb{P}_s ds. \]

\[ \leq 2^{p-1} \varepsilon \{ C' \mu + \max \{ 1, C'' \} (T - t_0) \lambda \phi(t) \} \]

\[ + 2^{p-1} \max \{ 1, C'' \} (T - t_0)^2 \int_{t_0}^T L(s, s, E \mathbb{P} x \mathbb{P}', E \mathbb{P} y \mathbb{P}') \mathbb{E} \mathbb{P} x - y \mathbb{P}_s ds. \]

There exists a constant

\[ h = \frac{1}{1 - 2^{p-1} \max \{ 1, C'' \} (T - t_0)^2 \int_{t_0}^T L(s, s, E \mathbb{P} x \mathbb{P}', E \mathbb{P} y \mathbb{P}') ds} > 0 \]

independent of \( \lambda \phi(t) \) such that
Thus, the system is Ulam-Hyers-Rassias stable in the mean. Hence the proof.

**Remark 2.5.4**

1. Under the assumption of Theorem 2.5.1, the system (2.2.1) and the inequality (2.5.1) are considered. One can repeat the same process to verify that the system (2.2.1) is Ulam-Hyers stable in the $p^{th}$ mean.

2. Under the assumption of Theorem 2.5.1, the system (2.2.1) and the inequality (2.5.2) are considered. One can repeat the same process to verify that the system (2.2.1) is generalized Ulam-Hyers-Rassias stable in the $p^{th}$ mean.
Chapter - 3
CHAPTER – 3
STABILITY RESULTS OF RANDOM IMPULSIVE SEMILINEAR DIFFERENTIAL EQUATIONS

In this chapter, the existence, uniqueness, continuous dependence, Ulam stabilities and exponential stability of random impulsive semilinear differential equations under sufficient conditions are found out. The results are obtained by using the contraction mapping principle. Finally an example is given to illustrate the applications of the abstract results.

3.1 Introduction

Impulsive differential equations are well known to model problems from many areas of science and engineering. There has been much research activity concerning the theory of impulsive differential equations (see [46, 67]). The impulses may exists at deterministic or random points. There are lot of papers which investigate the properties of deterministic impulses (see [5, 35, 46, 67] and the references therein).

When the impulses exist at random points, then the solutions of the differential equations is a stochastic process. It is very different from deterministic impulsive differential equations and also it is different from stochastic differential equations. Thus the random impulsive equations give more realistic than deterministic impulsive equations. There are few publications in this field, S.J. Wu and X.Z. Meng first brought forward
random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov’s direct function in [84]. S.J. Wu, and X.L. Guo et al, studied some qualitative properties of random impulses in [85-88]. In [8], A.Anguraj, S. Wu and A. Vinodkumar studied the existence and exponential stability for a random impulsive semilinear functional differential equations through the fixed point technique under non-uniqueness. The existence, uniqueness and stability results were discussed in [6] through Banach fixed point method for the system of differential equations with random impulsive effect. In [7,74-76] A.Vinodkumar and A. Anguraj studied the existence results for the random impulsive neutral functional differential equations and differential inclusions with delays. In [92], S.Zhang and J. Sun generalized the distribution of random impulses with the Erlang distribution.

The stabilities like continuous dependence, Hyers-Ulam stability, Hyers- Ulam- Rassias stability, exponential stability and asymptotic stability have attracted the attention of many mathematicians (see [14,37,66,73,78,79,80,82] and the references therein). In [78], J. Wang, L. Lv, and Y. Zhou have given the Ulam’s type stability and data dependence for fractional differential equations (FDEs). J. Wang, L. Lv, and Y. Zhou in [79] studied stability of FDEs using fixed point theorem
in a generalized complete metric space. In [80], J. Wang, Y. Zhou, and M. Fečkan studied Ulam’s stability for the nonlinear impulsive FDEs. Moreover, T.A.Burton and Bo Zhang [15], studied the existence and asymptotic stability through fixed point theory. Jiaowan Luo [41] studied the exponential stability and almost sure exponential stability in $p^{th}$ mean of mild solutions of stochastic differential equation by means of contraction mapping principle.

Motivated by the above mentioned works, the main purpose of this chapter is to investigate the random impulsive semilinear differential systems. The Lipschitz condition on the impulsive term is relaxed and under the assumption it is enough to be bounded. The results of Hyers-Ulam stability and Hyers-Ulam-Rassias stability are extended to fill the gap in abstract partial differential equation. The technique developed in [21,24,41,46,49,57,67,78,80,81,86] is utilized.

This chapter is organized as follows: In section 3.2, the notations, definitions, preliminary facts, existence and uniqueness theorem which are used throughout this chapter are recalled briefly. In section 3.3, the stability through continuous dependence on initial conditions of random impulsive semilinear differential systems is found out. The Hyers Ulam stability and Hyers Ulam-Rassias stability of the solutions of differential systems are investigated in section 3.4. The exponential stability of
solution of the random impulsive semilinear differential equations with delays is found out in section 3.5 by using contraction mapping principle. Finally in section 3.6, an example is presented to show our results.

3.2 Preliminaries

Let $X$ be a real separable Hilbert space and $\Omega$ a nonempty set. Assume that $\tau_k$ is a random variable defined from $\Omega$ to $D_k = (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. Furthermore, assume that $\tau_k$ follow Erlang distribution, where $k = 1, 2, \ldots$ and let $\tau_i$ and $\tau_j$ are independent with each other as $i \neq j$ for $i, j = 1, 2, \ldots$. For the sake of simplicity, denoted that $\mathcal{R}_\tau = [\tau, +\infty), \mathcal{R}^+ = [0, +\infty)$.

Consider the semilinear differential equations with random impulses of the form:

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
x'(t) &= Ax(t) + f(t, x_t), & t \neq \xi_k, \ t \geq t_0, \\
\xi_k &= b_k(t_k)x(\xi_k), & k = 1, 2, \ldots, \\
x_0 &= \varphi,
\end{array}
\right.
\tag{3.2.1}
\end{align*}
$$

where $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$ with domain $D(A) \subset X$; the functional $f : \mathcal{R}^+ \times C \rightarrow X$, $C = C([-r, 0], X)$ is the set of piecewise continuous functions mapping $[-r, 0]$ in to $X$ with some given $r > 0$; $x_t$ is a function when $t$ is fixed, defined by $x_t(s) = x(t + s)$ for all
Let $s \in [-r, 0]$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \cdots$, here $t_0 \in \mathbb{R}_{t}$ is arbitrary given real number. The impulse moments $\{\xi_k\}$ form a strictly increasing sequence, i.e., $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \lim_{k \to \infty} \xi_k = \infty$; $b_k : D_k \to X$ for each $k = 1, 2, \cdots$; $x(\xi_k^+) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm $P_x P = \sup_{t - r \leq s \leq t} |x(s)|$ for each $t$ satisfying $t \geq t_0$, $P \cdot P$ is any given norm in $X$; $\varphi$ is a function defined from $[-r, 0]$ to $X$.

Denote $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $F_t$ the $\sigma$-algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{F_t\})$ is a probability space. Let $L_2 = L_2(\Omega, F_t, X)$ denote the Hilbert space of all $F_t$-measurable square integrable random variables with values in $X$.

Assume that $T > t_0$ is any fixed time to be determined later and let $B$ denote the Banach space $B([t_0 - r, T], L_2)$, the family of all $F_t$-measurable, $C$-valued random variables $\psi$ with the norm

$$P_\psi P_B = \left( \sup_{t_0 \leq s \leq T} E P_\psi P_t \right)^{1/2}.$$

Let $L_2^0(\Omega, B)$ denote the family of all $F_0$-measurable, $B$-valued random variable $\varphi$. 

68
Definition 3.2.1

A semigroup \( \{S(t); t \geq t_0\} \) is said to be exponentially stable if there are positive constants \( M \geq 1 \) and \( \gamma > 0 \) such that \( \mathbb{P} S(t) \mathbb{P} \leq Me^{-\gamma(t-t_0)} \) for all \( t \geq t_0 \), where \( \mathbb{P} \mathbb{P} \) denotes the operator norm in \( \mathcal{L}(X) \) (The Banach algebra of bounded linear operators from \( X \) into \( X \)). A semigroup \( \{S(t), t \geq t_0\} \) is said to be uniformly bounded if \( \mathbb{P} S(t) \mathbb{P} \leq M \) for all \( t \geq t_0 \), where \( M \geq 1 \) is some constant. If \( M = 1 \), then the semigroup is said to be contraction semigroup.

Definition 3.2.2

For a given \( T \in (t_0, +\infty) \), a stochastic process \( \{x(t) \in \mathcal{B}, t_0 - r \leq t \leq T\} \) is called a mild solutions to equation (3.2.1) in \((\Omega, \mathcal{P}, \{\mathcal{F}_t\})\), if (i) \( x(t) \in \mathcal{B} \) is \( \mathcal{F}_t \)-adapted for \( t \geq t_0 \);

(ii) \( x(t_0 + s) = \varphi(s) \in L^0_2(\Omega, \mathcal{B}) \), when \( s \in [-r, 0] \), and

\[
x(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t-t_0) \varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{j}}^{\xi_{i}} S(t-s) f(s, x_s) ds
\]

\[
+ \int_{\xi_{k}}^{\xi_{k-1}} S(t-s) f(s, x_s) ds \right] I_{\{\xi_{k} < \xi_{k+1}\}}(t), \quad \text{cmt} \in [t_0, T],
\]

(3.2.2)

where \( \prod_{j=m}^{n}(\cdot) = 1 \) as \( m > n \), \( \prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) \ldots b_i(\tau_i) \), and \( I_{A}(\cdot) \) is the index function,
i.e., \[ I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases} \]

Now the following hypotheses are introduced, that used in the discussion:

\((H_1)\): The function \( f : [t_0, T] \times C \to X \) satisfy the Lipschitz conditions:
that is, there exit constants \( L_1 = L_1(T) \) and \( L_2 = L_2(T) > 0 \) for \( \zeta, \varsigma \in X \)
and for every \( t_0 \leq t \leq T \) such that

\[
E \| f(t, \zeta) - f(t, \varsigma) \| \leq L_1 E \| \zeta - \varsigma \|^2
\]

\[
E \| f(t, \zeta) \|^2 \leq L_2 (1 + E P \varsigma P).
\]

\((H_2)\): \( E \left( \max_{i,k} \left\{ \prod_{j=1}^{k} \| b_j(\tau_j) \| \right\} \right) \) is uniformly bounded, that is,
there is \( C > 0 \) such that

\[
E \left( \max_{i,k} \left\{ \prod_{j=1}^{k} \| b_j(\tau_j) \| \right\} \right) \leq C \quad \text{for all } \tau_j \in D_j, \ j = 1, 2, \ldots.
\]

**Theorem 3.2.1**

Let the hypotheses \((H_1) - (H_2)\) be hold. Then there exists a unique
(local) continuous mild solution to (3.2.1) for any initial value \((t_0, \varphi)\) with
\( t_0 \geq 0 \) and \( \varphi \in B \).

70
**Proof**

The nonlinear operator $\Phi : \mathcal{B} \to \mathcal{B}$ is defined as follows

$$(\Phi x)(t + t_0) = \varphi(t), t \in [-r, 0]$$

$$(\Phi x)(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0) \varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t - s) f(s, x_s) ds$$

$$+ \int_{\xi_k}^{t_0} S(t - s) f(s, x_s) ds \right] I_{[\xi_k, \xi_{k+1})}(t), 3cmt \in [t_0, T].$$

It is easy to prove the continuity of $\Phi$ in $[-r, T]$.

Now, to show that $\Phi$ maps $\mathcal{B}$ into itself.

$$P(\Phi x)(t) \leq \left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) P S(t - t_0) P \varphi(0) P$$

$$+ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) P \int_{\xi_{i-1}}^{\xi_i} P S(t - s) f(s, x_s) P ds \right]$$

$$+ \int_{\xi_k}^{t_0} P S(t - s) f(s, x_s) P ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2$$

$$\leq 2\left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_i(\tau_i) P S(t - t_0) P \varphi(0) P I_{[\xi_k, \xi_{k+1})}(t) \right]$$

$$+ \left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) P \int_{\xi_{i-1}}^{\xi_i} P S(t - s) P f(s, x_s) P ds \right]$$

$$+ \int_{\xi_k}^{t_0} P S(t - s) P f(s, x_s) P ds \right] I_{[\xi_k, \xi_{k+1})}(t) \right]^2 \right]$$
Thus, for all \( t \), therefore \( \Phi \) maps \( B \) into itself.

\[
E \mathbb{P}(\Phi x)(t) \mathbb{P} \leq 2M^2 E \max \{ \prod_{i=1}^{k} \mathbb{P}(b_i(\tau_i)) \mathbb{P} \} \mathbb{P} \mathbb{P} \varphi(0) \mathbb{P}
\]

\[
+ 2M^2 E [\max \{ 1, \prod_{j=i}^{k} b_j(\tau_j) \}^2 \times E(\int_{t_0}^{t} \| f(s, x_s) \| ds l_{(\xi_k, \xi_{k+1})}(t))^2
\]

\[
\leq 2M^2 C^2 E \mathbb{P} \varphi(0) \mathbb{P} + 2M^2 \max \{ 1, C^2 \} E(\int_{t_0}^{t} \| f(s, x_s) \| ds)^2
\]

\[
E \mathbb{P}(\Phi x) \mathbb{P} \leq 2M^2 C^2 E \mathbb{P} \varphi(0) \mathbb{P} + 2M^2 \max \{ 1, C^2 \} \int_{t_0}^{t} E \| f(s, x_s) \|^2 ds
\]

\[
E \mathbb{P}(\Phi x) \mathbb{P} \leq 2M^2 C^2 E \mathbb{P} \varphi(0) \mathbb{P} + 2M^2 \max \{ 1, C^2 \} (T - t_0) \int_{t_0}^{t} L_2 (1 + E \mathbb{P} x \mathbb{P} ) ds
\]

\[
\leq 2M^2 C^2 E \mathbb{P} \varphi(0) \mathbb{P} + 4M^2 \max \{ 1, C^2 \} (T - t_0)^2 L_2
\]

\[
+ 4M^2 \max \{ 1, C^2 \} (T - t_0) L_2 \int_{t_0}^{t} E \mathbb{P} x \mathbb{P} ds.
\]

Thus,

\[
\sup_{t \in [t_0, T]} E \mathbb{P}(\Phi x) \mathbb{P} \leq 2M^2 C^2 E \mathbb{P} \varphi(0) \mathbb{P} + 4M^2 \max \{ 1, C^2 \} (T - t_0)^2 L_2
\]

\[
+ 4M^2 \max \{ 1, C^2 \} (T - t_0) L_2 \int_{t_0}^{t} \sup_{s \in [t_0, T]} E \| x \|^2 ds
\]

\[
\leq 2M^2 C^2 E \mathbb{P} \varphi(0) \mathbb{P} + 4M^2 \max \{ 1, C^2 \} (T - t_0)^2 L_2
\]

\[
+ 4M^2 \max \{ 1, C^2 \} (T - t_0)^2 L_2 \sup_{t \in [t_0, T]} E \| x \|^2
\]

for all \( t \in [-r, T] \), therefore \( \Phi \) maps \( B \) into itself.
Now, to show that $\Phi$ is a contraction mapping

$$
\mathbb{P}(\Phi x(t) - (\Phi y)(t)) \leq \left[ \sum_{k=0}^{n} \left[ \sum_{i=1}^{k} \prod_{j=i}^{k} \mathbb{P}(b_j(\tau_j)) \right] \mathbb{P} \right]^{\sum_{i=1}^{k} \mathbb{P}((t-s)) \mathbb{P} f(s,x_s) - f(s,y_s) \mathbb{P} ds + \sum_{i=1}^{k} \mathbb{P}((t-s)) \mathbb{P} f(s,x_s) - f(s,y_s) \mathbb{P} ds \right]^{\sum_{i=1}^{k} \mathbb{P}((t-s)) \mathbb{P} f(s,x_s) - f(s,y_s) \mathbb{P} ds} 
$$

$$
\leq M^2 \left[ \max \left\{ 1, \prod_{j=i}^{k} \left\| b_j(\tau_j) \right\| \right\} \right]^{2} \times \left( \int_{t_0}^{t} \mathbb{P} f(s,x_s) - f(s,y_s) \mathbb{P} ds \right)^{2}
$$

$$
\mathbb{E} \mathbb{P}(\Phi x(t) - (\Phi y)(t)) \leq M^2 \max \{1, C^2\} (t-t_0) \int_{t_0}^{t} \mathbb{E} f(s,x_s) - f(s,y_s) \mathbb{P} ds 
$$

$$
\leq M^2 \max \{1, C^2\} (T-t_0) L_1 \int_{t_0}^{t} \mathbb{E} x - y \mathbb{P} ds.
$$

Taking supremum over $t$, then

$$
\mathbb{P}(\Phi x) - (\Phi y) \mathbb{P} \leq M^2 \max \{1, C^2\} (T-t_0)^2 L_1 \mathbb{P} x - y \mathbb{P}^2.
$$

Therefore,

$$
\mathbb{P}(\Phi x) - (\Phi y) \mathbb{P} \leq \Lambda(T) \mathbb{P} x - y \mathbb{P}^2,
$$

with $\Lambda(T) = M^2 \max \{1, C^2\} (T-t_0)^2 L_1$.

Take a suitable $0 < T_1 < T$ is taken by sufficient small such that $\Lambda(T_1) < 1$, and hence $\Phi$ is a contraction on $\mathcal{B}_{T_1}$ ( $\mathcal{B}_{T_1}$ denotes $\mathcal{B}$ with $T$ substituted by $T_1$ ). Thus, by the well-known Banach fixed point theorem a unique fixed point $x \in \mathcal{B}_{T_1}$ is obtained for operator $\Phi$, and hence $\Phi x = x$.
is a mild solution of the system (3.2.1). This procedure can be repeated to extend the solution to the entire interval $[-r, T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of mild solutions on the whole interval $[-r, T]$.

**Theorem 3.2.2**

Let $f : \mathbb{R}^+ \times C \to X$ satisfy the assumptions $(H_1) - (H_2)$. Then there exists a unique, global, continuous solution $x$ to (3.2.1) for any initial value $(t_0, \varphi)$ with $t_0 \geq 0$ and $\varphi \in B$.

**Proof**

Since $T$ is arbitrary in the proof of the previous theorem, this assertion follows immediately.

**3.3 Stability**

In this section, the stability of the system (3.2.1) through the continuous dependence of solutions on initial conditions are proved.

**Theorem 3.3.1**

Let $x(t)$ and $\bar{x}(t)$ be mild solutions of the system (3.2.1) with initial values $\varphi(0)$ and $\bar{\varphi}(0) \in B$ respectively. If the assumptions of Theorem 3.2.2 is satisfied, then the mild solution of the system (3.2.1) is stable in the mean square.
Proof

By the assumption, \( x \) and \( \tilde{x} \) are the two mild solutions of the system (3.2.1) for \( t \in [t_0, T] \). Then,

\[
x(t) - \tilde{x}(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0)[\varphi(0) - \varphi(0)] \\
+ \sum_{i=1}^{k} \prod_{j=1}^{i} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t - s)[f(s, x_s) - f(s, \tilde{x}_s)] ds \\
+ \int_{\xi_k}^{t} S(t - s)[f(s, x_s) - f(s, \tilde{x}_s)] ds \bigg] I_{[\xi_k, \xi_{k+1}]}(t).
\]

By using the hypotheses \((H_1) - (H_2)\),

\[
E P x - \tilde{x} P_t \leq 2 \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_i(\tau_i) P P S(t - t_0) P^2 E P \varphi(0) - \varphi(0) P^2 I_{[\xi_k, \xi_{k+1}]}(t) \\
+ 2E \left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_i(\tau_i) P \int_{\xi_{i-1}}^{\xi_i} P S(t - s) P P f(s, x_s) - f(s, \tilde{x}_s) P ds \right]^2 \\
+ \int_{\xi_k}^{t} P S(t - s) P P f(s, x_s) - f(s, \tilde{x}_s) P ds I_{[\xi_k, \xi_{k+1}]}(t) \right]^2 \\
\leq 2M^2 E \left\{ \max_k \left\{ \prod_{i=1}^{k} P b_i(\tau_i) P^2 \right\} \right\} E P \varphi(0) - \varphi(0) P^2 \\
+ 2M^2 E \left[ \max_{i,k} \left\{ \prod_{j=1}^{i} b_j(\tau_j) \right\} \right]^2 \\
\times E \left( \int_{t_0}^{t} P f(s, x_s) - f(s, \tilde{x}_s) P ds I_{[\xi_k, \xi_{k+1}]}(t) \right)^2
\]
By applying Gronwall’s inequality,

\[ \sup_{t \in [t_0, T]} \mathbb{E} |x - \bar{x}|^2 \leq 2M^2C^2 \mathbb{E} \phi(0) - \overline{\phi(0)}^2 \]

\[ + 2M^2 \max\{1, C^2\} (T - t_0) \int_{t_0}^T \mathbb{E} f(s, x_s) - f(s, \bar{x}_s) \overline{\phi} \, ds \]

\[ + 2M^2 \max\{1, C^2\} (T - t_0) L \int_{t_0}^T \sup_{s \in [t_0, t]} \mathbb{E} |x| \overline{\phi} \, ds. \]

By applying Gronwall’s inequality,

\[ \sup_{t \in [t_0, T]} \mathbb{E} |x - \bar{x}|^2 \leq 2M^2C^2 \mathbb{E} \phi(0) - \overline{\phi(0)}^2 \]

\[ \times \exp(2M^2 \max\{1, C^2\} (T - t_0)^2 L) \]

\[ \leq \Gamma \mathbb{E} \phi(0) - \overline{\phi(0)}^2, \]

where, \( \Gamma = 2M^2C^2 \exp(2M^2 \max\{1, C^2\} (T - t_0)^2 L) \).

Now given \( \varepsilon > 0 \), choose \( \delta = \frac{\varepsilon}{\Gamma} \) such that \( \mathbb{E} \phi(0) - \overline{\phi(0)}^2 < \delta \). Then

\[ \sup_{t \in [t_0, T]} \mathbb{E} |x - \bar{x}|^2 \leq \varepsilon. \]

Thus, it is apparent that the difference between the mild solution \( x(t) \) and \( \bar{x}(t) \) in the interval \([t_0, T]\) is small provided the change in the initial point \((t_0, \phi(0))\) as well as in the function \( f(t, x_s) \) do not exceed prescribed amounts. This completes the proof.
3.4 Ulam-Hyers-Rassias Stability

In this section, the Ulam-Hyers stability of random impulsive semilinear differential equations (3.2.1) is found out.

Let \( \varepsilon > 0 \), \( \mu \geq 0 \) and \( \phi : [t_0, T] \to \mathbb{R}^+ \) be a piecewise continuous function. Consider the following inequalities:

\[
\begin{align*}
E P x'(t) - Ax(t) - f(t, x_t) P^2 &\leq \varepsilon, \quad t \neq \xi_k, \quad t \geq t_0. \\
E P x(\xi_k) - b_k(\tau_k) x(\xi_k^-) P^2 &\leq \varepsilon, \quad k = 1, 2, \ldots.
\end{align*}
\] (3.4.1)

\[
\begin{align*}
E P x'(t) - Ax(t) - f(t, x_t) P^2 &\leq \phi(t), \quad t \neq \xi_k, \quad t \geq t_0. \\
E P x(\xi_k) - b_k(\tau_k) x(\xi_k^-) P^2 &\leq \mu, \quad k = 1, 2, \ldots.
\end{align*}
\] (3.4.2)

\[
\begin{align*}
E P x'(t) - Ax(t) - f(t, x_t) P^2 &\leq \varepsilon \phi(t), \quad t \neq \xi_k, \quad t \geq t_0. \\
E P x(\xi_k) - b_k(\tau_k) x(\xi_k^-) P^2 &\leq \varepsilon \mu, \quad k = 1, 2, \ldots.
\end{align*}
\] (3.4.3)

**Definition 3.4.1**

The system (3.2.1) is Ulam-Hyers stable in the mean square if there exists a real number \( \kappa > 0 \) such that for each \( \varepsilon > 0 \) and for each mild solution \( x \in B \) of the inequality (3.4.1) there exists a mild solution \( y \in B \) of the system (3.2.1) with \( E P x(t) - y(t) P^2 \leq \kappa \varepsilon, \quad t \in [t_0, T] \).

**Definition 3.4.2**

The system (3.2.1) is generalized Ulam-Hyers stable in the mean square if there exists a real number \( \eta \in B, \eta(0) = 0 \) such that for each
mild solution $x \in \mathcal{B}$ of the inequality (3.4.1) there exists a mild solution

$y \in \mathcal{B}$ of the system (3.2.1) with $E \mathcal{P} x(t) - y(t) \leq \eta(\epsilon), \ t \in [t_0, T]$.

**Definition 3.4.3**

The system (3.2.1) is Ulam- Hyers- Rassias stable in the mean square with respect to $(\phi, \mu)$ if there exists a real number $\zeta > 0$ such that for each $\epsilon > 0$ and for each mild solution $x \in \mathcal{B}$ of the inequality (3.4.3) there exists a mild solution $y \in \mathcal{B}$ of the system (3.2.1) with

$$E \mathcal{P} x(t) - y(t) \leq \zeta \epsilon (\phi(t) + \mu), \ t \in [t_0, T].$$

**Definition 3.4.4**

The system (3.2.1) is generalized Ulam- Hyers- Rassias stable in the mean square with respect to $(\phi, \mu)$ if there exists a real number $\zeta > 0$ such that for each mild solution $x \in \mathcal{B}$ of the inequality (3.4.2) there exists a mild solution $y \in \mathcal{B}$ of the system (3.2.1) with

$$E \mathcal{P} x(t) - y(t) \leq \zeta (\phi(t) + \mu), \ t \in [t_0, T].$$

**Remark 3.4.1**

It is clear that

1. Definition (3.4.1) $\Rightarrow$ Definition (3.4.2)

2. Definition (3.4.3) $\Rightarrow$ Definition (3.4.4)

3. Definition (3.4.3) for $\phi(t) = \mu = 1$ $\Rightarrow$ Definition (3.4.1).
Remark 3.4.2

A function \( x \in B \) is a mild solution of the inequality (3.4.3) if and only if there exists a function \( h \in B \) and the sequence \( h_k, k = 1, 2, \ldots \) (which depend on \( x \)) such that

(i): \( E Ph(t) \leq \epsilon \phi(t), t \in [t_0, T] \) and \( E Ph_k \leq \epsilon \mu, k = 1, 2, \ldots; \)

(ii): \( x'(t) = Ax(t) + f(t, x(t)) + h(t), \quad t \neq \xi_k, \quad t \geq t_0; \)

(iii): \( x(\xi_k) = b_k(\tau_k) x(\xi_k^-) + h_k, \quad k = 1, 2, \ldots. \)

One can have similar remarks for the inequalities (3.4.1) and (3.4.2).

Remark 3.4.3

If \( x \in B \) is a mild solution of the inequality (3.4.3) then \( x \) is a mild solution of the following integral inequality

\[
E P x(t) - \sum_{k=0}^{\infty} \left[ \prod_{j=1}^{k} b_j(\tau_j) S(t-t_0) \varphi(0) + \sum_{i=1}^{k} \sum_{j=i}^{k} b_j(\tau_j) \int_{\xi_j}^{\xi_{j-1}} S(t-s) f(s, x_s) \, ds \right] \\
+ \int_{\xi_k}^{t} S(t-s) f(s, x_s) \, ds \right] I_{[\xi_k, \xi_{k+1})} (t) \mathcal{P} \\
\leq 2M^2 \mathcal{C}^2 \mu + \max \{1, \mathcal{C}^2\} \int_{t_0}^{t} \phi(s) \, ds, \quad t \in [t_0, T].
\]

From the Remark 3.4.2,

\[
x'(t) = Ax(t) + f(t, x(t)) + h(t), \quad t \neq \xi_k, \quad t \geq t_0. \\
x(\xi_k) = b_k(\tau_k) x(\xi_k^-) + h_k, \quad k = 1, 2, \ldots.
\]

(3.4.4)
Then

\[ x(t + t_0) = \varphi(t), \text{ for } t \in [-r, 0], \]

\[ x(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0) \varphi(0) + \prod_{i=1}^{k} b_i(\tau_i) S(t - s) h_i \]

\[ + \sum_{i=1}^{+\infty} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t - s) f(s, x_s) ds \]

\[ + \int_{\xi_{k}}^{t} S(t - s) f(s, x_s) ds + \sum_{i=1}^{+\infty} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t - s) h(s) ds \]

\[ + \int_{\xi_{k}}^{t} S(t - s) h(s) ds I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [t_0, T]. \]

Therefore,

\[ E \mathbb{P} [x(t) - \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0) \varphi(0) + \prod_{i=1}^{k} b_i(\tau_i) \int_{\xi_{i-1}}^{\xi_i} S(t - s) f(s, x_s) ds \]

\[ + \int_{\xi_{k}}^{t} S(t - s) f(s, x_s) ds I_{[\xi_k, \xi_{k+1})}(t) \mathbb{P}^2 \]

\[ = E \mathbb{P} \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0) h_i + \prod_{i=1}^{k} b_i(\tau_i) \int_{\xi_{i-1}}^{\xi_i} S(t - s) h(s) ds \]

\[ + \int_{\xi_{k}}^{t} S(t - s) h(s) ds I_{[\xi_k, \xi_{k+1})}(t) \mathbb{P}^2 \]

\[ \leq 2M^2 E \left\{ \max_{k} \prod_{i=1}^{k} P b_i(\tau_i) \mathbb{P} \right\} E \mathbb{P} h_1 \mathbb{P} \]

\[ + 2M^2 E \left[ \max_{i,k} \left\{ \prod_{j=1}^{k} P b_i(\tau_j) \mathbb{P} \right\}^2 (T - t_0) \int_{0}^{t} E \mathbb{P} h(s) \mathbb{P} ds \right] \]

\[ \leq 2M^2 \varepsilon \{ C^2 \mu + \max\{1, C^2\} (T - t_0) \int_{0}^{t} \phi(s) ds \} . \]
There are similar remarks for the mild solutions of the inequalities (3.4.1) and (3.4.2).

Now, Ulam-Hyers-Rassias stability results are given in this section.

**Theorem 3.4.1**

Assumption $(H_1)$ and $(H_2)$ hold. Suppose there exists $\lambda > 0$ such that $\int_0^t \phi(s)ds \leq \lambda \phi(t)$, for each $t \in [t_0, T]$, where $\phi: [t_0, T] \rightarrow \mathbb{R}^+$ is a continuous nondecreasing function. Then the system (3.2.1) is Ulam-Hyers-Rassias stable in the mean square.

**Proof**

Let $x \in \mathcal{B}$ be a mild solution of the inequality (3.4.3). By Theorem 4.2.1 there exist a unique mild solution $y$ of the random impulsive differential system

\[
\begin{cases}
y'(t) = Ay(t) + f(t, y_t), & t \neq \xi_k, \quad t \geq t_0 \\
y(\xi_k) = b_k(\tau_k) y(\xi_k^-), & k = 1, 2, ... \\
y_0 = \varphi.
\end{cases}
\]  

(3.4.5)

Then

\[
y(t + t_0) = \varphi(t), \quad \text{for } t \in [-r, 0],
\]

\[
y(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0) \varphi(0) + \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t - s) f(s, y_s) ds
\]

81
By differential inequality (3.4.3),

\[
E\mathbb{P}(t) - \sum_{k=0}^{+\infty} \prod_{i=1}^k b_i(\tau_i) S(t - t_0) \phi(0) + \sum_{j=1}^{+\infty} \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{j-1}}^{\xi_j} S(t - s) f(s, x_s) ds
\]

\[
+ \int_{\xi_{k}}^{T} S(t - s) f(s, x_s) ds \right] \mathbb{I}_{\{\xi_k, \xi_{k+1}\}}(t) \mathbb{P}
\]

\[
= E\mathbb{P} \sum_{k=0}^{+\infty} \prod_{i=1}^k b_i(\tau_i) h_i + \sum_{j=1}^{+\infty} \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{j-1}}^{\xi_j} S(t - s) h(s) ds
\]

\[
+ \int_{\xi_{k}}^{T} S(t - s) h(s) ds \right] \mathbb{I}_{\{\xi_k, \xi_{k+1}\}}(t) \mathbb{P}
\]

\[
\leq 2M^2 E \left\{ \max \left\{ \prod_{i=1}^k P b_i(\tau_i) \mathbb{P} \right\} \right\} E\mathbb{P} h_i \mathbb{P}
\]

\[
+ 2M^2 E \left\{ \max \left\{ 1, \prod_{i=1}^k P b_i(\tau_i) \mathbb{P} \right\} \right\} (T - t_0) \times \varepsilon \int_{0}^{+\infty} \phi(s) ds
\]

\[
\leq 2M^2 \varepsilon \left\{ C^2 \mu + \max \{ 1, C^2 \} (T - t_0) \lambda \phi(t) \right\}, \quad t \in [t_0, T].
\]

Hence for each \( t \in [t_0, T] \),

\[
E\mathbb{P}(t) - y(t) \mathbb{P} = E\mathbb{P}(t) - \sum_{k=0}^{+\infty} \prod_{i=1}^k b_i(\tau_i) S(t - t_0) \phi(0)
\]

\[
+ \sum_{j=1}^{+\infty} \prod_{j=1}^k b_j(\tau_j) \int_{\xi_{j-1}}^{\xi_j} S(t - s) f(s, x_s) ds
\]

\[
+ \int_{\xi_{k}}^{T} S(t - s) f(s, x_s) ds \right] \mathbb{I}_{\{\xi_k, \xi_{k+1}\}}(t) \mathbb{P}
\]

\[
E\mathbb{P} x - y \mathbb{P} \leq 2E\mathbb{P}(t) - \sum_{k=0}^{+\infty} \prod_{i=1}^k b_i(\tau_i) S(t - t_0) \phi(0)
\]

82
\[ E \mathbb{P}x - y \mathbb{P}_t^P \leq 2E \mathbb{P}x(t) - \sum_{k=0}^{+\infty} \left( \prod_{i=1}^{k} b_i(\tau_i) S(t-t_0) \phi(0) \right) \]

\[ + \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) f(s,x_s) ds \]

\[ + \int_{\xi_k}^{t} S(t-s) f(s,x_s) ds I_{(\xi_k, \xi_{k+1})}(t) \mathbb{P} \]

\[ + 2E \mathbb{P} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) \{ f(s,x_s) - f(x,y_s) \} ds \]

\[ + \int_{\xi_k}^{t} S(t-s) \{ f(s,x_s) - f(x,y_s) \} ds I_{(\xi_k, \xi_{k+1})}(t) \mathbb{P} \]

\[ \leq 4M^2 \varepsilon \{ C^2 \mu + \max \{ 1, C^2 \} (T - t_0) \lambda \phi(t) \} \]

\[ + 2M^2 \max \{ 1, C^2 \} (T - t_0) L_1 \int_{t_0}^{t} E \mathbb{P}x - y \mathbb{P}_s^P ds. \]

\[ \sup_{t \in [t_0, T]} E \mathbb{P}x - y \mathbb{P}_t^P \leq 4M^2 \varepsilon \{ C^2 \mu + \max \{ 1, C^2 \} (T - t_0) \lambda \phi(t) \} \]

\[ + 2M^2 \max \{ 1, C^2 \} (T - t_0) L_1 \int_{t_0}^{t} \sup_{s \in [t_0, t]} E \mathbb{P}x - y \mathbb{P}_s^P ds. \]

\[ \leq 4M^2 \varepsilon \{ C^2 \mu + \max \{ 1, C^2 \} (T - t_0) \lambda \phi(t) \} \]

\[ + 2M^2 \max \{ 1, C^2 \} L_1 (T - t_0)^2 \sup_{t \in [t_0, T]} E \mathbb{P}x - y \mathbb{P}_t^P. \]

There exists a constant \( h = \frac{1}{1 - 2M^2 \max \{ 1, C^2 \} L_1 (T - t_0)^2} > 0 \) independent of \( \lambda \phi(t) \) such that
Thus, the system (3.2.1) is Ulam - Hyers-Rassias stable in the mean square.

Hence the proof.

Remark 3.4.4

1. Under the assumption of Theorem 3.4.1, the system (3.2.1) and the inequality (3.4.1) are considered. One can repeat the same process to verify that the system (3.2.1) is Ulam - Hyers stable in the mean square.

2. Under the assumption of Theorem 3.4.1, the system (3.2.1) and the inequality (3.4.2) are considered. One can repeat the same process to verify that the system (3.2.1) is generalized Ulam - Hyers-Rassias stable in the mean square.

3.5 Exponential Stability

In this section, the exponential stability of the second moment of a mild solution of the system (3.2.1) is found out.

For any $F_t$–adapted process $\Psi(t):[-r,\infty) \to \mathbb{R}$ is almost surely continuous in $t$. For the purposes of stability, assume that $f(t,0) \equiv 0$ for any $t \geq t_0$, so that the system (3.2.1) admits a trivial solution. Moreover,
\( \Psi(t) = \varphi(t-t_0) \) for \( t \in [t_0-r, t_0] \) and \( e^{\alpha(t-t_0)}E \mathbb{P} \Psi \mathbb{P} \to 0 \) as \( t \to \infty \), where \( \alpha \) is a positive constant such that \( 0 < \alpha < \gamma \).

**Definition 3.5.1**

Equation (3.2.1) is said to be exponentially stable in the quadratic mean if there exists positive constants \( \hat{C} \) and \( \hat{\lambda} > 0 \) such that

\[
E \| x(t) \|^2 \leq \hat{C} E \| \varphi \|^2 e^{-\hat{\lambda}(t-t_0)}, \quad t \geq t_0.
\]

Now, assume the following additional assumptions:

\( (H'_1) \): The function \( f : \mathbb{R}^+ \times \mathbb{C} \to X \) satisfy the Lipschitz conditions: that is, there exits a constants \( L > 0 \) for \( \zeta, \varsigma \in X \) and for every \( t \geq t_0 \) such that

\[
E \| f(t, \zeta) - f(t, \varsigma) \|^2 \leq L E \| \zeta - \varsigma \|^2
\]

\( (H_3) \): \( \mathbb{P} S(t) \mathbb{P} \leq M e^{-\gamma(t-t_0)}, \quad t \geq t_0 \), where \( M \geq 1 \) and \( \gamma > 0 \).

**Theorem 3.5.1**

Let the assumptions \( (H'_1) \) and \( (H_3) \) hold. Then the system (3.2.1) is exponentially stable in the quadratic mean, if the following inequalities hold

\[
\max \{ 1, C^2 \} M^2 L / (\gamma - \alpha) < \gamma \quad \text{and} \quad MC \geq \frac{1}{\sqrt{2}}. \quad (3.5.1)
\]

98
Proof

Define the nonlinear operator $\Phi: B \to B$ by $(\Phi x)(t + t_0) = \varphi(t)$ for $[-r, 0]$ and for $t \geq t_0$,

$$(\Phi x)(t) = \sum_{k=0}^{+\infty} \left( \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \right)_{\xi_i-1}^{\xi_i} S(t - s) f(s, x_s) ds$$

$$+ \int_{\xi_k}^{t} S(t - s) f(s, x_s) ds \bigg] I_{[\xi_k, \xi_{k+1}]}(t).$$

Now, to verify the continuity of $\Phi$ in the quadratic mean on $[t_0, \infty)$.

Let $x \in B$, $t_1 \geq t_0$ and $|h|$ be sufficiently small, then by using hypotheses and condition (3.5.1),

$$(\Phi x)(t_1 + h) - (\Phi x)(t_1)$$

$$= \sum_{k=0}^{+\infty} \left( \prod_{i=1}^{k} b_i(\tau_i) S(t_1 + h - t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \right)_{\xi_i-1}^{\xi_i} S(t_1 + h - s) f(s, x_s) ds$$

$$+ \int_{\xi_k}^{t_1 + h} S(t_1 + h - s) f(s, x_s) ds \bigg] I_{[\xi_k, \xi_{k+1}]}(t_1 + h)$$

$$- \sum_{k=0}^{+\infty} \left( \prod_{i=1}^{k} b_i(\tau_i) S(t_1 - t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \right)_{\xi_i-1}^{\xi_i} S(t_1 - s) f(s, x_s) ds$$

$$+ \int_{\xi_k}^{t_1} S(t_1 - s) f(s, x_s) ds \bigg] I_{[\xi_k, \xi_{k+1}]}(t_1).$$

86
Thus,

\[(\Phi x)(t_1 + h) - \Phi x(t_1)\]

\begin{align*}
&= \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t_1 + h - t_0) \varphi(0) + \sum_{j=1}^{k} \prod_{i=1}^{k} b_j(\tau_j) \int_{\xi_{ji-1}}^{\xi_{ji}} S(t_1 + h - s) f(s, x_s) ds \\
&\quad + \int_{\xi_{k}}^{\xi_{k+1}} S(t_1 + h - s) f(s, x_s) ds \left[ I_{[\xi_{k} \xi_{k+1}]}(t_1 + h) - I_{[\xi_{k} \xi_{k+1}]}(t_1) \right] \\
&+ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) (S(t_1 + h - t_0) - S(t_1 - t_0)) \varphi(0) \\
&+ \left[ \sum_{j=1}^{k} \prod_{i=1}^{k} b_j(\tau_j) \int_{\xi_{ji-1}}^{\xi_{ji}} (S(t_1 + h - s) - S(t_1 - s)) f(s, x_s) ds \\
&+ \int_{t_1}^{t_1+\xi_{k}} (S(t_1 + h - s) - S(t_1 - s)) f(s, x_s) ds \right] I_{[\xi_{k} \xi_{k+1}]}(t_1 + h) \\
&+ \left. \int_{t_1}^{t_1+\xi_{k}+\xi_{k+1}} S(t_1 + h - s) f(s, x_s) ds \right] I_{[\xi_{k} \xi_{k+1}]}(t_1 + h). \end{align*}

\[E P(\Phi x)(t_1 + h) - (\Phi x)(t_1) P \leq 2E P I_1 \bar{P} + 2E P I_2 \bar{P}, \quad (3.5.2)\]

where

\[I_1 = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t_1 + h - t_0) \varphi(0) + \sum_{j=1}^{k} \prod_{i=1}^{k} b_j(\tau_j) \int_{\xi_{ji-1}}^{\xi_{ji}} S(t_1 + h - s) f(s, x_s) ds \\
+ \int_{\xi_{k}}^{\xi_{k+1}} S(t_1 + h - s) f(s, x_s) ds \left[ I_{[\xi_{k} \xi_{k+1}]}(t_1 + h) - I_{[\xi_{k} \xi_{k+1}]}(t_1) \right] \]

\[I_2 = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) (S(t_1 + h - t_0) - S(t_1 - t_0)) \varphi(0) \]

87
\[\begin{align*}
\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{j-1}}^{\xi_j} (S(t_1 + h - s) - S(t_1 - s)) f(s, x_s) ds
\end{align*}\]

\[\begin{align*}
+ \int_{\xi_k}^{\xi_{k+1}} (S(t_1 + h - s) - S(t_1 - s)) f(s, x_s) ds
\end{align*}\]

\[\begin{align*}
+ \int_{t_1}^{t_1 + h} S(t_1 + h - s) f(s, x_s) ds] I_{[\xi_{k-1}, \xi_k]}(t_1 + h).
\end{align*}\]

Now,

\[\begin{align*}
E P I_1 P \leq \big(2 E \max \left\{ \prod_{i=1}^{k} PB_i(\tau_i) P \right\} \big) P S(t_1 + h - t_0) P E \Phi(0) P
\end{align*}\]

\[\begin{align*}
\times (I_{[\xi_{k-1}, \xi_k]}(t_1) - I_{[\xi_{k-1}, \xi_k]}(t_1))^2
\end{align*}\]

\[\begin{align*}
+ 2E \max_{i,k} \left\{ 1, \prod_{j=i}^{k} b_j(\tau_j) \right\}^2 \big( E \sum_{i=k}^{\infty} \int_{0}^{t_1} P S(t_1 + h - s) P P f(s, x_s) P ds
\end{align*}\]

\[\begin{align*}
\times (I_{[\xi_{k-1}, \xi_k]}(t_1) - I_{[\xi_{k-1}, \xi_k]}(t_1))^2
\end{align*}\]

\[\begin{align*}
\leq [2 C^2 M^2 e^{-2\gamma (t_1 + h - t_0)} E \Phi(0) P
\end{align*}\]

\[\begin{align*}
+ 2 \max \{1, C^2\} (t_1 - t_0) E \int_{0}^{t_1} e^{-2\gamma (t_1 + h - s)} P f(s, x_s) P ds
\end{align*}\]

\[\begin{align*}
\times E (I_{[\xi_{k-1}, \xi_k]}(t_1) - I_{[\xi_{k-1}, \xi_k]}(t_1)) \to 0 \text{ as } h \to 0.
\end{align*}\]

and,

\[\begin{align*}
E P I_2 P \leq 3 E \max \left\{ \prod_{i=1}^{k} PB_i(\tau_i) P \right\} \big) P S(t_1 + h - t_0) - S(t - t_0) P E \Phi(0) P
\end{align*}\]
Thus, the right hand side of (3.5.2) tends to 0 as \( h \to 0 \). Hence, \( \Phi \) is continuous in the quadratic mean on \( [t_0, \infty) \).

Next, to show that \( \Phi \) maps \( \mathcal{B} \) into itself.
\[ e^{a(t-t_0)} E^\Phi(\Phi x) \leq 2e^{a(t-t_0)} E[\max_i \prod_j b_j(\tau_j) P_i] P S(t-t_0) \leq E^\Phi(0) \]

\[ +2e^{a(t-t_0)} E[\max_i \prod_j b_j(\tau_j)]^2 E[\int_0^t P S(t-s) \| f(s,x) \| ds I_{[\tau_k,\tau_{k+1}]}(t)]^2 \]

\[ e^{a(t-t_0)} E^\Phi(\Phi x) \leq I_3 + I_4, \quad \text{(3.5.3)} \]

where, \[ I_3 = 2E[\max_i \prod_j b_j(\tau_j)] e^{a(t-t_0)} P S(t-t_0) E^\Phi \]

\[ \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty, \quad \text{(3.5.4)} \]

and

\[ I_4 = 2e^{a(t-t_0)} E[\max_i \prod_j b_j(\tau_j)]^2 E[\int_0^t P S(t-s) \| f(s,x) \| ds I_{[\tau_k,\tau_{k+1}]}(t)]^2 \]

\[ \leq 2 \max \{1, C^2\} e^{a(t-t_0)} E[\int_0^t M e^{-\gamma(t-s)} \| f(s,x) \| ds]^2 \]

\[ = 2 \max \{1, C^2\} e^{a(t-t_0)} M^2 E[\int_0^t e^{-\frac{\gamma(t-s)}{2}} \cdot e^{-\frac{\gamma(t-s)}{2}} \| f(s,x) \| ds]^2 \]

\[ \leq 2 \max \{1, C^2\} e^{a(t-t_0)} M^2 \int_0^t e^{-\gamma(t-s)} ds \int_0^t e^{-\gamma(t-s)} E[\| f(s,x) \|^2 ds] \]

\[ \leq \frac{2 \max \{1, C^2\} M^2 e^{a(t-t_0)}}{\gamma} \left( \int_0^t e^{-\gamma(t-s)} E[\| f(s,x) \|^2 ds] \right) \]
For any $x(t) \in B$ and $\varepsilon > 0$, there exists a $t_1 > 0$ such that

$e^{\alpha(s-t_0)}E\|x\|^2 < \varepsilon$ for $t \geq t_1$. Thus from (3.3.5)

$I_4 \leq \frac{2\max\{1,C^2\}M^2 e^{-(\gamma-\alpha)(t-t_0)}}{\gamma} \int_{t_0}^{t} e^{(\gamma-\alpha)(s-t_0)} e^{\alpha(s-t_0)} E\|x\|^2 ds + \frac{2\max\{1,C^2\}M^2 e^{-(\gamma-\alpha)(t-t_0)}}{\gamma} \int_{t_1}^{t} e^{(\gamma-\alpha)(s-t_0)} e^{\alpha(s-t_0)} E\|x\|^2 ds$

$\leq \frac{2\max\{1,C^2\}M^2 e^{-(\gamma-\alpha)(t-t_0)}}{\gamma} \int_{t_0}^{t} e^{(\gamma-\alpha)(s-t_0)} e^{\alpha(s-t_0)} E\|x\|^2 ds + \frac{2\max\{1,C^2\}M^2 \left(\frac{1}{\gamma-\alpha}\right)\varepsilon. (3.5.6)$

As $e^{-(\gamma-\alpha)(t-t_0)} \to 0$ as $t \to \infty$ and the condition (3.5.1), there exists $t_2 \geq t_1$ such that for any $t \geq t_2$,

$\frac{2\max\{1,C^2\}M^2 e^{-(\gamma-\alpha)(t-t_0)}}{\gamma} \int_{t_0}^{t} e^{(\gamma-\alpha)(s-t_0)} e^{\alpha(s-t_0)} E\|x\|^2 ds$

$\leq \varepsilon - \frac{2\max\{1,C^2\}M^2 \left(\frac{1}{\gamma-\alpha}\right)\varepsilon. (3.5.7)$
So from (3.5.6) and (3.5.7), for any $t \geq t_2$, $I_4 < \epsilon$ is obtained, that is to say,

$$I_4 \to 0 \text{ as } t \to \infty.$$  \hspace{1cm} (3.5.8)

Thus, from (3.5.3), (3.5.4) and (3.5.8), it know that

$$e^{\alpha(t-t_0)} E \mathcal{P}(\Phi x) \mathcal{P}_t \to 0 \text{ as } t \to \infty.$$  \hspace{1cm} (3.5.9)

Thus $\Phi$ maps $\mathcal{B}$ into itself.

Now, to show that $\Phi$ is a contraction mapping. For any $x, y \in \mathcal{B}$, the following is obtained that

$$E \mathcal{P}(\Phi x) - (\Phi y) \mathcal{P}_t^2 \leq E \left[ \max \left\{ 1, \prod_{j=1}^{k} \| b_j(\tau_j) \| \right\} \right]^2$$

$$\times E \left( \int_0^t \mathcal{P} \mathcal{S}(t-s) \mathcal{P} \left\| f(s,x_s) - f(s,y_s) \right\| ds \right)^2$$

$$\leq \max \left\{ 1, C^2 \right\} M^2 E \left( \int_0^t e^{-\gamma(t-s)} \left\| f(s,x_s) - f(s,y_s) \right\| ds \right)^2$$

$$= \max \left\{ 1, C^2 \right\} M^2 E \left( \int_0^t e^{-\frac{\gamma(t-s)}{2}} \cdot e^{-\frac{\gamma(t-s)}{2}} \left\| f(s,x_s) - f(s,y_s) \right\| ds \right)^2$$

$$\leq \frac{\max \left\{ 1, C^2 \right\} M^2 L}{\gamma} \int_0^t e^{-\gamma(t-s)} E \mathcal{P} x - y \mathcal{P}_s^2 ds$$

$$\sup_{t \in [0,T]} E \mathcal{P}(\Phi x) - (\Phi y) \mathcal{P}_t^2 \leq \frac{\max \left\{ 1, C^2 \right\} M^2 L}{\gamma} \int_0^1 e^{-\gamma(t-s)} \sup_{s \in [0,T]} E \mathcal{P} x - y \mathcal{P}_s^2 ds$$
Thus by (3.3.1), this shows that $\Phi$ is a contraction mapping. Hence $\Phi$ has a unique fixed point $x(t) \in \mathcal{B}$, which is the solution of (3.2.1) with $x(t + t_0) = \varphi(t)$ for $[-r, 0]$ and $e^{\alpha(t-t_0)} E \mathbb{P}_x \mathcal{P}_t \to 0$ as $t \to \infty$. This completes the proof.

3.6 An Application

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$.

$$
\begin{cases}
  u_t(x,t) = u_{xx}(x,t) + \int_{-r}^t \mu(\theta)u(t+\theta,x)d\theta, & t \neq \xi_k, t \geq \tau, \\
  u(x,\xi_k) = q(k)\tau_k u(x,\xi_k^-) & \text{a.s.} & x \in \Omega, \quad (3.6.1) \\
  u(x,t) = \varphi(x,t) & \text{a.s.} & x \in \Omega, -r \leq t \leq 0, \\
  u(x,t) = 0 & \text{a.s.} & x \in \partial \Omega.
\end{cases}
$$

Let $X = L^2(\Omega)$, and $\tau_k$ be a random variable defined on $D_k = (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$ and $\mu: [-r, 0] \to \mathbb{R}$ is a positive function. Furthermore, assume that $\tau_k$ follow Erlang distribution, where $k = 1, 2, \ldots$ and $\tau_i$ and $\tau_j$ are independent with each other as $i \neq j$ for $i, j = 1, 2, \ldots$; $q$ is a function of $k$; $\xi_0 = t_0$; $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$ and $t_0 \in \mathbb{R}^+$ is an arbitrarily given real number.
Define $A$ an operator on $X$ by $Au = \frac{\partial^2 u}{\partial x^2}$ with the domain $D(A) = \{ u \in X \mid u$ and $\frac{\partial u}{\partial x}$ are absolutely continuous, $\frac{\partial^2 u}{\partial x^2} \in X$, $u = 0$ on $\partial \Omega \}$. It is well known that $A$ generates a strongly continuous semigroup $S(t)$ which is compact, analytic and self adjoint. Moreover, the operator $A$ can be expressed as

$$Au = \sum_{n=1}^{\infty} n^2 < u, u_n > u_n , u \in D(A),$$

where $u_n(\zeta) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\zeta)$, $n = 1, 2, \ldots$, is the orthonormal set of eigenvectors of $A$ and for every $u \in X$, $S(t)u = \sum_{n=1}^{\infty} \exp(-n^2t) < u, u_n > u_n$, which satisfies $\mathbb{P}S(t) \mathbb{P} \leq \exp(-\pi^2(t-t_0))$, $t \geq t_0$. Hence $S(t)$ is a contraction semigroup.

Now, assume the following conditions hold:

1. The function $\mu(\cdot)$ is continuous in $\Re$ with

$$\int_{-\rho}^{0} \mu(\theta)^2 d\theta < \infty.$$

2. $E\left[ \max_{i,k} \left\{ \prod_{j=i}^{k} \left\| q(j)(\tau_j) \right\|^2 \right\} \right] < \infty.$
Assuming that conditions (i) and (ii) are verified, then the problem (3.6.1) can be modeled as the abstract random impulsive differential equation (3.2.1) by defining

\[ f(t,u_t) = \int_{-r}^{t} \mu(\theta)u(t + \theta, x)d\theta, \text{ and } b_k(\tau_k) = q(k)\tau_k. \]

**Proposition 3.6.1**

Assume that the hypotheses \((H_1)-(H_2)\) hold, then the system (3.6.1) has a unique, global mild solution \(u\).

**Proof**

Condition (i) implies that \((H_1)\) holds with \(L_1 = \int_{-r}^{0} \mu^2(\theta)d\theta\) and \((H_2)\) follow from conditions (ii).

The next results are consequences of Theorems 3.3.1, 3.4.1 and 3.5.1 respectively.

**Proposition 3.6.2**

Let the hypotheses \((H_1)\) and \((H_2)\) be hold. Then the mild solution \(u\) of the system (3.6.1) is stable in the mean square.

**Proposition 3.6.3**

Let the hypotheses \((H_1)\) and \((H_2)\) be hold. Suppose there exists \(\lambda > 0\) such that
\[ \int_{t_0}^{t} \phi(s) ds \leq \lambda \phi(t), \text{ for each } t \in [t_0, T], \]

where \( \phi: [t_0, T] \to \mathbb{R}^+ \) is a continuous nondecreasing function. Then the system (3.6.1) is generalized Ulam - Hyers - Rassias stable in the mean square.

**Proposition 3.6.4**

Assume that the hypotheses \( (H'_1) \) and \( (H_3) \) hold, then the mild solution \( u \) for the system (3.6.1) is exponentially stable in the quadratic mean provided,

\[ \max\{1, C^2\} L / (\pi^2 - \alpha) < \pi^2 \]

is satisfied.

**Proof**

Condition (i) implies that \( (H_1) \) holds with \( L = \int_{-\pi}^{\pi} \mu^2(\theta) d\theta \) and \( (H_2) \) follow from conditions (ii).
Chapter - 4
In this Chapter, the existence, uniqueness and stability via continuous dependence of mild solution of neutral partial differential equations with random impulses are investigated under sufficient conditions via fixed point theory.

4.1 Introduction

Neutral differential equations arise in many area of science and engineering have received much attention in the last decades. The ordinary neutral differential equation is very extensive to study the theory of aeroelasticity [44] and the lossless transmission lines [27] and the references therein. Neutral partial differential equations with delays are motivated from stabilization of lumped control systems, theory of heat conduction in materials ([31,35] and the references therein). E. Hernández and Donal O’regan [32], studied some neutral partial differential equations by assuming some temporal and spatial regularity type condition for the function \( t \to g(t,x) \), which have not been considerer in the literature.
Recently impulsive differential equations are well to model problems see [46, 67]. There is much notice in the field of fixed impulsive type equations ([5, 35] and the references therein). When the impulses are exist at random, the solutions of the equation behaves as a stochastic process. It is quite different from deterministic impulsive differential equations and stochastic differential equations (SDEs). R. Iwankiewicz and S.R.K. Nielsen [39], investigated dynamic response of non-linear systems to Poisson distributed random impulses. S.J. Wu and X.Z. Meng [84] first gave the general random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov’s direct method. S.J. Wu, X.L. Guo et al.[85,86,87], have studied some qualitative properties of differential equations with random impulses. In [8], A.Anguraj, S. Wu and A. Vinodkumar studied the existence and exponential stability for a random impulsive semilinear functional differential equations through the fixed point technique under non-uniqueness. The existence, uniqueness and stability results were discussed in [6] through Banach fixed point method for the system of differential equations with random impulsive effect. A. Anguraj and A.Vinodkumar [7], studied the existence results for the random impulsive neutral functional differential equations with delays. In [74], A.Vinodkumar and A. Anguraj studied existence results of random impulsive neutral non-autonomous differential inclusions with delays via
Dhage’s fixed point theorem. In [75] A.Vinodkumar, random impulsive semilinear functional differential inclusions were studied using the Martelli fixed point theorem and the fixed point theorem due to Covitz and Nadler. In [18], Y.K. Chang and W.T. Li generalized the distribution of random impulses with the Erlang distribution. Further refer [14, 21, 57]. Motivated by the above mentioned works, the main purpose of this chapter is to investigate the random impulsive neutral partial differential equations (RINDEs) to fill the gap in the above works. Aeroelasticity and the lossless transmission lines can also be modelled in the form of RINDEs.

This chapter is organized as follows. In section 4.2, the notations, definitions, lemmas and preliminaries which are used throughout this chapter are recalled briefly. In section 4.3, the existence and uniqueness of the RINDEs by relaxing the linear growth conditions are found out. In section 4.4, the stability through continuous dependence on the initial values of the RINDEs is found out. Finally in section 4.5, an example is presented to illustrate the results.

4.2 Preliminaries

Let $X$ be a real separable Hilbert space and $\Omega$ a nonempty set. Assume that $\tau_k$ is a random variable defined from $\Omega$ to $D_k = (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. Furthermore, assume that $\tau_k$ follow
Erlang distribution, where \( k = 1, 2, \ldots \) and let \( \tau_i \) and \( \tau_j \) are independent with each other as \( i \neq j \) for \( i, j = 1, 2, \ldots \). For the sake of simplicity, denoting the \( \mathbb{R}^+ = [0, +\infty) \).

Now consider the neutral partial differential equations with random impulses of the form:

\[
\begin{split}
\frac{d}{dt}[x(t) + g(t, x_i)] &= Ax(t) + f(t, x_i), \quad t \neq \xi_k, \quad t \geq 0, \\
x(\xi_k) &= b_k(\tau_k) x(\xi_k^-), \quad k = 1, 2, \ldots, \\
x_{0} &= \varphi,
\end{split}
\]

(4.2.1)

(4.2.2)

(4.2.3)

where \( A \) is the infinitesimal generator of an analytic semigroup of bounded linear operators \( \{S(t); t \geq 0\} \) with \( D(A) \subset X \). If \( S(t) \) is uniformly bounded analytic semigroup such that \( 0 \in \rho(A) \), then it is possible to define the fractional power \( A^\eta \), for \( 0 < \eta \leq 1 \), as a closed linear operator with dense domain \( D(A^\eta) \) in \( X \). If \( X_\eta \) represents the space \( D(A^\eta) \) endowed with norm \( \| \cdot \| \). Then,

**Lemma 4.2.1 [57]**

*Assume that the following conditions hold:*

(i): For \( 0 < \eta \leq 1 \), \( X_\eta \) is a Banach space.
(ii): For $0 < \eta \leq \beta \leq 1$, the embedding $X_\beta \rightarrow X_\eta$ is continuous.

(iii) There exists a constant $C_\eta > 0$ depending on $0 < \eta \leq 1$ such that

$$\mathbb{P} A^t S(t) \mathbb{P} \leq \frac{C_\eta}{t^{2\eta}}, \quad t > 0.$$ 

Now make the system (4.2.1) – (4.2.3) precious: The functional $g: \mathbb{R}^+ \times \mathcal{C} \rightarrow X$; $f: \mathbb{R}^+ \times \mathcal{C} \rightarrow X$, $\mathcal{C} = \mathcal{C}((-\infty,0],X_\eta)$ is the set of piecewise continuous functions with left-hand limit $\varphi$ from $(-\infty,0]$ into $X_\eta$. The phase space $\mathcal{E}'((-\infty,0],X_\eta)$ is assumed to be equipped with the norm $\mathbb{P}\varphi \mathbb{P} = \sup_{-\infty < \theta \leq 0} |\varphi(\theta)|$. $x_i$ is a function defined by $x_i(s) = x(t+s)$ for all $s \in (-\infty,0]$ and fixed $t$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1,2,\ldots$, here $t_0 \in \mathbb{R}^+$ is arbitrary given real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \ldots < \lim_{k \rightarrow \infty} \xi_k = \infty$; $b_k: D_k \rightarrow X$ for each $k = 1,2,\ldots$; $x(\xi_k) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm $\mathbb{P}x \mathbb{P} = \sup_{-\infty < s \leq t} |x(s)|$ for each $t$ satisfying $t \geq 0$ and $T \in \mathbb{R}^+$ is a given number, $\mathbb{P} \cdot \mathbb{P}$ is any given norm in $X_\eta$.

Denote $\{B_i, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_i \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the $\sigma$-algebra generated by $\{B_i, t \geq 0\}$. Then $\Omega, \mathbb{P}, \{\mathcal{F}_t\}$ is a probability space. Let $L^2 = L^2(\Omega, \mathcal{F}_t, X)$.
denote the Hilbert space of all $F_t$-measurable square integrable random variables with values in $X$.

Let $B_T$ denote the Banach space $B_T([t_0,T],L_2)$, the family of all $F_t$-measurable, $C$-valued random variables $\psi$ with the norm

$$P_{\psi}^T = \sup_{t\in[t_0,T]} E|\psi|^2.$$

Let $L^0_2(\Omega,B_T)$ denote the family of all $F_0$-measurable, $B_T$-valued random variable $\phi$.

**Definition 4.2.1**

A semigroup $\{S(t), t \geq t_0\}$ is said to be uniformly bounded if $PS(t) \leq M$ for all $t \geq t_0$, where $M \geq 1$ is some constant. If $M=1$, then the semigroup is said to be contraction semigroup.

**Definition 4.2.2**

For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in B_T, -\infty < t \leq T\}$ is called a mild solution to system (4.2.1)-(4.2.3) in $(\Omega, P, \{F_t\})$, if

(i) $x(t) \in B_T$ is $F_t$-adapted;

(ii) $x(t_0 + s) = \phi(s)$ when $s \in (-\infty, 0]$, and
\[
x(t) = \sum_{k=0}^{+\infty} [\prod_{i=1}^{k} b_i(\tau_i) S(t - t_0)[\varphi(0) + g(0, \varphi)] - \prod_{i=1}^{k} b_i(\tau_i) g(t, x_i)]
\]

\[
-\left[ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \right] \int_{\xi_{i-1}}^{\xi_i} AS(t - s) g(s, x_s) ds + \int_{\xi_{k}}^{t} AS(t - s) g(s, x_s) ds
\]

\[
+ \left[ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \right] \int_{\xi_{i-1}}^{\xi_i} S(t - s) f(s, x_s) ds + \int_{\xi_{k}}^{t} S(t - s) f(s, x_s) ds] I_{\xi_{k} \gamma_{k+1}}(t), \quad t \in [t_0, T],
\]

where \( \prod_{j=m}^{n}(i) = 1 \) as \( m > n \), \( \prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})\ldots b_1(\tau_1) \), and \( I_A(\cdot) \) is the index function,

\[
i.e., \quad I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}
\]

Now introduce the following hypotheses are used in the discussion:

**Hypotheses**

\((H_1)\): The function \( f : [t_0, T] \times C \to X \) satisfies the Lipschitz condition, that is there exists a constant \( L_f = L_f(T) > 0 \) such that

\[
EPf(t, x_t) - f(t, y_t) \leq L_f EPx - y P_t \quad \text{for } x, y \in C, t \in [t_0, T].
\]

\((H_2)\): The mapping \( g : [t_0, T] \times C \to X \) satisfies that there exists a number \( \eta \in [0,1] \) such that for any \( x, y \in C, t \in [t_0, T] \) and \( g(t, x_t) \in D(A^\eta) \) and

\[
EP A^\eta g(t, x_t) - A^\eta g(t, y_t) \leq L_g EPx - y P_t, \quad L_g > 0.
\]
(H₃): For all \( t \in [t₀, T] \), it follows that \( f(t,0), A^n g(t,0) \in L^1 \) such that

\[
E \| f(t,0) \| \leq κ_f,
\]

\[
E \| A^n g(t,0) \| \leq κ_g,
\]

where \( κ_f, κ_g > 0 \) are constants.

(H₄): The condition \( E \left\{ \max_{i,k} \left\{ \prod_{j=1}^{k} \| b_j(τ_j) \| \right\} \right\} \) is uniformly bounded. That is, there is a constant \( C > 0 \) such that

\[
E \left\{ \max_{i,k} \left\{ \prod_{j=1}^{k} \| b_j(τ_j) \| \right\} \right\} \leq C \quad \text{for all } τ_j ∈ D_j, \; j = 1, 2, \ldots.
\]

### 4.3 Existence and uniqueness

In this section, discuss the existence and uniqueness of the mild solution of the system (4.2.1)-(4.2.3).

**Theorem 4.3.1**

Let the hypotheses (H₁) – (H₄) be hold. Then there exists a unique (local) continuous mild solution to (4.2.1)-(4.2.3) for any initial value \((t₀, φ)\) with \( t₀ ≥ 0 \) and \( φ ∈ B_τ \).

**Proof**

Let \( T \) be an arbitrary number \( t₀ < T < +∞ \). In order to apply the contraction principle, the nonlinear operator \( Φ : B_τ → B_τ \) is defined as follows \((Φx)(t) = φ(t − t₀), \; \text{for } t ∈ (−∞, t₀] \) and for \( t ∈ [t₀, T] \)
\[(\Phi x)(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)[\varphi(0) + g(0, \varphi)] - \prod_{i=1}^{k} b_i(\tau_i) g(t, x_i)\]

\[-\left[ \sum_{j=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} AS(t-s)g(s, x_s)ds + \int_{\xi_k}^{t} AS(t-s)g(s, x_s)ds \right]\]

\[+ \left[ \sum_{j=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s, x_s)ds + \int_{\xi_k}^{t} S(t-s)f(s, x_s)ds \right] I_{[\xi_k, \xi_{k+1}]}(t).\]

It is easy to prove the continuity of \(\Phi\). Now, to show that \(\Phi\) maps \(\mathcal{B}_T\) into itself.

\[\mathbb{P}(\Phi x)(t) \mathbb{P} \leq 4\left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_i(\tau_i) \mathbb{P} S(t-t_0) \mathbb{P} \varphi(0) + g(0, \varphi) \mathbb{P} I_{[\xi_k, \xi_{k+1}]}(t) \right]\]

\[+ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_i(\tau_i) \mathbb{P} A^{-\eta} \mathbb{P} A^\eta g(t, x_i) \mathbb{P} I_{[\xi_k, \xi_{k+1}]}(t)\]

\[+ \left[ \max_{i,k} \left\{ 1, \prod_{j=i}^{k} \| b_j(\tau_j) \| \right\} \right]^2 \left( \int_{t_0}^{t} P A^{-\eta} S(t-s) A^\eta g(s, x_s) P d\mathbb{P} I_{[\xi_k, \xi_{k+1}]}(t) \right)^2\]

\[+ \left[ \max_{i,k} \left\{ 1, \prod_{j=i}^{k} \| b_j(\tau_j) \| \right\} \right]^2 \left( \int_{t_0}^{t} P S(t-s) P f(s, x_s) P d\mathbb{P} I_{[\xi_k, \xi_{k+1}]}(t) \right)^2 \]

\[\leq 8 M^2 \left[ \max_{k} \left\{ \prod_{i=1}^{k} P b_i(\tau_i) \right\} \right] \left[ \mathbb{P} \varphi(0) \mathbb{P} + \mathbb{P} g(0, \varphi) \mathbb{P} \right]\]

\[+ 8 \left[ \max_{k} \left\{ \prod_{i=1}^{k} P b_i(\tau_i) \right\} \right] \mathbb{P} A^{-\eta} \mathbb{P} \left[ \mathbb{P} A^\eta g(t, x_i) - A^\eta g(t, 0) \mathbb{P} + \mathbb{P} A^\eta g(t, 0) \mathbb{P} \right]\]
By Lemma 4.2.1, \((H_2)\) and \((H_3)\) the following relation holds:

\[
E \mathcal{P} \mathcal{A} S(t-s)g(s,x_s) \mathcal{P} = E \mathcal{P} \mathcal{A}^{1-\eta} S(t-s)A^\eta g(s,x_s) \mathcal{P}
\]

\[
\leq 2 \mathcal{P} \mathcal{A}^{1-\eta} S(t-s) \mathcal{P} [ E \mathcal{P} A^\eta g(s,x_s) - A^\eta g(s,0) \mathcal{P} + E \mathcal{P} A^\eta g(s,0) \mathcal{P} ]
\]

\[
\leq \frac{2C_{1-\eta}}{(t-s)^{2(1-\eta)}} [L_g E P x \mathcal{P} + k_g ]. \quad (4.3.1)
\]

Then, \(E \mathcal{P} (\Phi x) \mathcal{P} \leq 8M^2C^2E[\mathcal{P}\phi(0) \mathcal{P} + \mathcal{P}g(0,\phi) \mathcal{P}]\)

\[
+8C^2 \mathcal{P} A^{-\eta} \mathcal{P} [L_g E P x \mathcal{P} + \kappa_g ]
\]

\[
+8 \max\{1,C^2\} T \int_{0}^{t} \frac{C_{1-\eta}}{(t-s)^{2(1-\eta)}} [L_g E P x \mathcal{P} + k_g ] ds
\]

\[
+8M^2 \max\{1,C^2\} T \int_{0}^{t} [L_f E P x \mathcal{P} + \kappa_f ] ds.
\]

Taking supremum over \(t\),

\[
\mathcal{P} \Phi x \mathcal{P}_{B_t} \leq c_1 + c_2 \mathcal{P} x \mathcal{P}_{B_t},
\]

where \(c_i, i = 1,2\), are constants. Hence \(\Phi\) is bounded.
Now, to show $\Phi$ is a contraction mapping. For any $x, y \in B_T$, then

$$P(\Phi x)(t) - (\Phi y)(t) P^k$$

$$\leq 3\max\left\{ \prod_{i=1}^{k} \left\| b_j(\tau_i) \right\| \right\} P \left[ A^{-\eta} P A^{\eta} g(t, x_i) - A^{\eta} g(t, y_i) P I_{[\xi_k, \xi_{k+1}]}(t) \right]$$

$$+ 3\left[ \max_{i, k} \left\{ \prod_{j=i}^{k} \left\| b_j(\tau_j) \right\| \right\} \right]^2$$

$$\times \left[ \int_{0}^{t} P A^{-\eta} S(t-s) [A^{\eta} g(t, x_s) - A^{\eta} g(t, y_s)] P ds I_{[\xi_k, \xi_{k+1}]}(t) \right]^2$$

$$E P(\Phi x) - (\Phi y) P^k \leq 3C^2 P A^{-\eta} P^k E P A^{\eta} g(t, x_i) - A^{\eta} g(t, y_i) P^k$$

$$+ 3\max\{1, C^2\} (t-t_0) \int_{0}^{t} \frac{C_{1-\eta}}{(t-s)^{2(1-\eta)}} E P A^{\eta} g(t, x_s) - A^{\eta} g(t, y_s) P^k ds$$

$$+ 3\max\{1, C^2\} (t-t_0) M^2 \int_{0}^{t} E P f(s, x_s) - f(s, y_s) P^k ds.$$

Thus,

$$E P(\Phi x) - (\Phi y) P^k$$

$$\leq \left[ (3C^2 P A^{-\eta} P^k + 3\max\{1, C^2\} \frac{C_{1-\eta} T^{2\eta}}{2\eta - 1}) L_g + 3\max\{1, C^2\} M^2 T^2 L_f \right] E P x - y P^k.$$

Hence,

$$P(\Phi x) - (\Phi y) P^k_{B_T} \leq \Lambda(T) P x - y P^k_{B_T},$$

with

$$\Lambda(T) = \left[ (3C^2 P A^{-\eta} P^k + 3\max\{1, C^2\} \frac{C_{1-\eta} T^{2\eta}}{2\eta - 1}) L_g + 3\max\{1, C^2\} M^2 T^2 L_f \right]$$

107
Then take a suitable $0 < T_1 < T$ sufficiently small such that $\Lambda(T_1) < 1$, and hence $\Phi$ is a contraction on $B_{T_1}$ ($B_{T_1}$ denotes $B_T$ with $T$ substituted by $T_1$). Thus, by the well-known Banach fixed point theorem a unique fixed point $x \in B_{T_1}$ is obtained for operator $\Phi$, and hence $\Phi x = x$ is a mild solution of (4.2.1)–(4.2.3). This procedure can be repeated to extend the solution to the entire interval $(-\infty, T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of mild solutions on the whole interval $(-\infty, T]$.

**Theorem 4.3.2**

Let $g : \mathbb{R}^+ \times \mathbb{C} \to X$ and $f : \mathbb{R}^+ \times \mathbb{C} \to X$ satisfy the assumptions $(H_1)$–$(H_4)$. Then there exists a unique, global, continuous solution $x$ to (4.2.1)–(4.2.3) for any initial value $(t_0, \varphi)$ with $t_0 \geq 0$ and $\varphi \in B_T$.

**Proof**

Since $T$ is arbitrary in the proof of the previous theorem, this assertion follows immediately.

**4.4 Stability**

In this section, the stability through continuous dependence of solutions on initial condition is investigated.
**Definition 4.4.1**

A mild solution \( x(t) \) of the system (4.2.1)-(4.2.2) with initial value \( \phi \) satisfies (4.2.3) is said to be stable in the mean square if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
E \mathbb{P} x(t) - \hat{x}(t) \leq \varepsilon \quad \text{whenever} \quad E \mathbb{P} \phi - \hat{\phi} \mathbb{P} < \delta, \quad \text{for all} \quad t \in [t_0, T]. \tag{4.4.1}
\]

where \( \hat{x}(t) \) is another mild solution of the system (4.2.1)–(4.2.2) with initial value \( \hat{\phi} \) defined in (4.2.3).

**Theorem 4.4.1**

Let \( x(t) \) and \( y(t) \) be mild solutions of the system (4.2.1)–(4.2.3) with initial values \( \varphi_1 \) and \( \varphi_2 \) respectively. If the assumptions of Theorem 4.3.2 are satisfied, then the mild solution of the system (4.2.1)–(4.2.3) is stable in the mean square.

**Proof**

By the assumptions, \( x(t) \) and \( y(t) \) are two mild solutions of equations (4.2.1)–(4.2.3) with initial values \( \varphi_1 \) and \( \varphi_2 \) respectively, then

\[
x(t) - y(t) = \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i) S(t - t_0) [\varphi_1 - \varphi_2] + [g(0, \varphi_1) - g(0, \varphi_2)]
\]
\[-\prod_{i=1}^{k} b_i(\tau_i) [g(t,x_i) - g(t,y_i)]\]

\[-\left\{ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\tau_{i-1}}^{\tau_i} AS(t-s)[g(s,x_s) - g(s,y_s)]ds \right\}\]

\[+ \int_{\tau_k}^{t} AS(t-s)[g(s,x_s) - g(s,y_s)]ds\]

\[+ \left\{ \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\tau_{i-1}}^{\tau_i} S(t-s)[f(s,x_s) - f(s,y_s)]ds \right\}\]

\[+ \int_{\tau_k}^{t} S(t-s)[f(s,x_s) - f(s,y_s)]ds\right\}]I_{(\tau_k, \tau_{k+1})}(t)\]

Then, 

\[E PX(t) - y(t) \leq 8M^2 \left[ \max_{k} \left\{ \prod_{i=1}^{k} Pb_i(\tau_i) \right\} \right] E[\mathbb{P} \varphi_1 - \varphi_2 \mathbb{P}^*] \]

\[+ \mathbb{P} A^{-\eta} \mathbb{P} E \mathbb{P} A^{g}(0,\varphi_1) - A^{g}(0,\varphi_2) \mathbb{P}^* \]

\[+4 \left[ \max_{k} \left\{ \prod_{i=1}^{k} Pb_i(\tau_i) \right\} \right] \mathbb{P} A^{-\eta} \mathbb{P} E[\mathbb{P} A^{g}(t,x_i) - A^{g}(t,y_i) \mathbb{P}^*] \]

\[+4 \left[ \max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(\tau_j)\| \right\} \right]^2 \]

\[\times (t-t_0) \int_{t_0}^{t} E \mathbb{P} A^{g}(s,x_s) - A^{g}(s,y_s) \mathbb{P}^* ds \]

\[+4M^2 \left[ \max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(\tau_j)\| \right\} \right]^2 (t-t_0) \int_{t_0}^{t} E \mathbb{P} f(s,x_s) - f(s,y_s) \mathbb{P}^* ds\]
Therefore,

\[
\sup_{t \in [0,T]} E P x - y \bar{P}_t \leq 4 M^2 \left( C^2 + \frac{1}{2} \right) L_g \sup_{t \in [0,T]} E P x - y \bar{P}_t
\]

\[
+ 4 C^2 P A^{-\eta} \bar{P}_s \sup_{s \in [0,t]} E P x - y \bar{P}_s ds
\]

\[
+ 4 \max \{ 1, C^2 \} T \int_0^t \frac{C_{1-q}}{(t-s)^{2(1-q)}} L_g \sup_{s \in [t_0,t]} E P x - y \bar{P}_s ds
\]

\[
+ 4 M^2 \max \{ 1, C^2 \} T \int_0^t L_f \sup_{s \in [t_0,t]} E P x - y \bar{P}_s ds.
\]

Therefore,

\[
\sup_{t \in [0,T]} E P x - y \bar{P}_t \leq \bar{Y} E P \phi_1 - \phi_2 \bar{P}_t,
\]

where,

\[
\bar{Y} = \frac{8 M^2 C^2 [1 + P A^{-\eta} \bar{P} L_g]}{1 - [4 C^2 P A^{-\eta} \bar{P} + 4 \max \{ 1, C^2 \} C_{1-q} T^{2\eta} L_g] + 4 M^2 \max \{ 1, C^2 \} T^2 L_f}
\]

Now given \( \varepsilon > 0 \), choose \( \delta = \frac{\varepsilon}{\bar{Y}} \) such that \( E P \phi_1 - \phi_2 \bar{P} < \delta \). Then

\[
P x - y \bar{P}_{\bar{Y} \varepsilon} \leq \varepsilon.
\]

This completes the proof.

4.5 An Example

Now conclude this work with an example of the form

\[
\begin{cases}
\frac{\partial}{\partial t} [u(t, x) + \int_0^\pi b(y, x)u(tsint, y)dy] = \frac{\partial^2}{\partial x^2} u(t, x) + H(t, u(tsint, x)), & t \neq \xi_k, \\
u(x, \xi_k) = q(k)\pi u(x, \xi_k), & t = \xi_k, \\
u(t, 0) = u(t, \pi) = 0 \\
u(x, t) = \Phi(x, t) & 0 \leq x \leq \pi, \ -\infty < t \leq 0, \ t \geq 0.
\end{cases}
\]

(4.5.1)
Let $X = L^2([0, \pi])$, and $\tau_k$ be a random variable defined on $D_k \equiv (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. Furthermore, assume that $\tau_k$ follow Erlang distribution, where $k = 1, 2, \ldots$ and $\tau_i$ and $\tau_j$ are independent with each other as $i \neq j$ for $i, j = 1, 2, \ldots$; $q$ is a function of $k$; $\xi_0 = t_0$; $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$ and $t_0 \in \mathbb{R}^+$ is an arbitrary given real number.

Define $A$ an operator on $X$ by $Au = \frac{\partial^2 u}{\partial x^2}$ with the domain

$$D(A) = \{u \in X \mid u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in X, u(0) = u(\pi) = 0\}.$$

It is well known that $A$ generates a strongly continuous semigroup $S(t)$ which is compact, analytic and self-adjoint. Moreover, the operator $A$ can be expressed as $Au = \sum_{n=1}^{\infty} n^2 < u, u_n > u_n$, $u \in D(A)$, where

$$u_n(\zeta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\zeta), \ n = 1, 2, \ldots,$$

is the orthonormal set of eigenvectors of $A$. Then the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}u = \sum_{n=1}^{\infty} n < u, u_n > u_n \text{ on the space } D[A^{\frac{1}{2}}] = \{u \in X; \sum_{n=1}^{\infty} n < u, u_n > u_n \in X\}.$$
This satisfies $PS(t)P \leq \exp(-\pi^2(t-t_0))$, $t \geq t_0$. Hence $S(t)$ is a contraction semigroup. In particular,

$$PA^{\frac{1}{2}}P = \frac{1}{\Gamma^{\frac{1}{2}}} \int_{0}^{\infty} t^{\frac{1}{2}-1} PS(t)Pdt < \frac{1}{\pi}.$$ 

Now, assume that the following conditions hold:

(i): The function $b$ is measurable and

$$\int_{0}^{\pi} \int_{0}^{\pi} b^2(y, x)dydx < \infty.$$ 

(ii): The function $\frac{\partial}{\partial t} b(y, x)$ is measurable $b(y, 0) = b(y, \pi) = 0$ and let

$$L_{b} = [\int_{0}^{\pi} \int_{0}^{\pi} (\frac{\partial}{\partial t} b(y, x))^2 dydx]^{\frac{1}{2}} < \infty.$$ 

(iii): For the function $H : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ the following three conditions are satisfied.

1. For each $t \in [0, \infty)$, $H(t, \cdot)$ is continuous.

2. For each $u \in X$, $H(\cdot, u)$ is measurable.

3. There are positive functions $h_1, h_2 : [0, \infty) \to \mathbb{R}^+$ such that

$$|H(t, u)| \leq h_1(t) + h_2 |u|, \ (t, u) \in [0, \infty) \times X;$$

113
Assuming that conditions (i) and (iv) are verified. Under these conditions, defining the \( f, g \) and \( b_k \) by

\[
f(t, x_i) = H(t, u(tsint, x)), \quad g(t, x_i) = \int_0^\pi b(y, x)u(tsint, y)dy
\]

\[
b_k(\tau_k) = q(k)\tau_k.
\]

Then the problem (4.5.1) can be modeled as the abstract random impulsive neutral functional differential equation of the form (4.2.1) – (4.2.3).

The next results are consequence of Theorem 4.3.2 and Theorem 4.4.1 respectively.

**Proposition 4.5.1**

Let the hypotheses \((H_1) – (H_4)\) be hold. Then there exists a unique global mild solution \( u \) of the system (4.5.1) provided.

**Proposition 4.5.2**

Let the conditions of Proposition (4.5.1) be hold. Then the mild solution \( u \) of the system (4.5.1) is stable in the mean square.
Chapter - 5
In this chapter, the existence, uniqueness, stability through continuous dependence on initial conditions, Hyers-ulam stability and Hyers-Ulam-Rassias stability results for random impulsive fractional differential systems by relaxing the linear growth conditions are investigated. Finally examples are given to illustrate the applications of the abstract results.

5.1 Introduction

Impulsive differential equations are suitable mathematical model to simulate the evolution of large classes of real processes. These processes are subjected to short temporary perturbations. The duration of these perturbations are negligible compared to the duration of whole process. These perturbations occurs in the form of impulses (see the monographs [46, 67]). When the impulses are exist at random it affect the nature of the differential system. There are few results that have been discussed. R. Iwankievicz and S.R.K. Nielsen [39], investigated the dynamic response of non-linear systems to Poisson distributed random impulses. A. Anguraj and A.Vinodkumar studied the existence, uniqueness and stability results of random impulsive semilinear differential systems [6].
J. M. Sanz-Serna and A.M. Stuart [68] first brought dissipative differential equations with random impulses and used Markov chains to simulate such systems. On further read on random impulsive type differential equations (refer [8,71,83,86,87] and the references therein).

Recently, fractional differential equations (FDEs) and impulsive fractional differential equations (IFDEs) have proved to be valuable tool in the modeling of many phenomena in various fields of science and engineering. Similarly, the stabilities like continuous dependence, Hyers-Ulam stability, Hyers-Ulam-Rassias stability, local stability and Mittag-Leffler stability for FDEs and IFDEs have attracted the attention of many mathematicians (see [22, 44, 51, 52, 54, 58, 63] and the references therein). In [78], J. Wang, L. Lv, and Y. Zhou have given the Ulam type stability and data dependence for FDEs. J. Wang, L. Lv, and Y. Zhou [79] studied stability of FDEs using fixed point theorem in a generalized complete metric space. In [80], J. Wang, Y. Zhou, and M. Fečkan studied Ulam’s stability for the nonlinear IFDEs. Michal Feckan, Yong Zhou and JinRong Wang, proved on the concept and existence of solution for IFDEs [53]. For more details on FDE and its stability concepts see [1,2,13, 17,37,48,55,56,59,73].

Motivated by the above mentioned works, the main purpose of this chapter is to investigate of the random impulsive fractional differential
systems. The Lipschitz condition on the impulsive term is relaxed and under our assumption it is enough to be bounded. To best of the knowledge there is no paper which investigate the random impulsive type fractional differential equations. For Hyers-Ulam stability and Hyers-Ulam- Rassias stability, the technique from [78, 80] is utilized.

This Chapter is organized as follows: In section 5.2, the notations, definitions, lemmas and preliminaries which are used throughout this chapter are recalled briefly. In section 5.3, the existence and uniqueness of solutions of random impulsive fractional differential systems by relaxing the linear growth condition are investigated. In section 5.4, the stability through continuous dependence on initial conditions of random impulsive fractional differential systems is found out. The Hyers Ulam stability and Hyers Ulam-Rassias stability of the solutions of random impulsive fractional differential systems are investigated in section 5.5 and finally in section 5.6 examples are given to illustrate the theoretical results.

5.2 Preliminaries

Let \( P \) denote Euclidean norm in \( \mathbb{R}^n \). Let \( \mathbb{R}^n \) be the \( n \) – dimensional Euclidean space and \( \Omega \) a nonempty set. Assume that \( \tau_k \) is a random variable defined from \( \Omega \) to \( D_k^{\text{def.}} = (0,d_k) \) for all \( k = 1,2,\ldots \), where
\( 0 < d_k < +\infty \). Furthermore, assume that \( \tau_i \) and \( \tau_j \) are independent with each other as \( i \neq j \) for \( i, j = 1, 2, \ldots \). Let \( \tau, T \in \mathcal{R} \) be two constants satisfying \( \tau < T \). For the sake of simplicity, denoting that \( \mathcal{R}_\tau = [\tau, T] \).

Consider the fractional differential system with random impulses of the form:

\[
\begin{align*}
\frac{d}{\alpha} x(t) &= f(t, x(t)), \quad t \neq \xi_k, \quad t \geq \tau, \\
x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, \ldots, \\
x(0) &= x_0.
\end{align*}
\]

where the function \( f : \mathcal{R} \times \mathcal{R}^n \rightarrow \mathcal{R}^n \); \( b_k : D_k \rightarrow \mathcal{R}^{n \times n} \) is a matrix-valued function for each \( k = 1, 2, \ldots \); \( \xi_0 = t_0 \) and \( \xi_k = \xi_{k-1} + \tau_k \) for \( k = 1, 2, \ldots \), here \( t_0 \in \mathcal{R}_\tau \) is arbitrary real number. Obviously,

\[
t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots \quad \text{and} \quad x(\xi_k^-) = \lim_{t \rightarrow \xi_k} x(t) \quad \text{according to their paths with the norm} \quad \mathbb{P} x \mathbb{P}^t = \sup_{t \in [\tau, s]} |x(s)| \quad \text{for each} \quad t \text{ satisfying} \quad \tau \leq t \leq T.
\]

Let us denote \( \{B_i, t \geq 0\} \) be the simple counting process generated by \( \{\xi_n\} \), that is, \( \{B_i \geq n\} = \{\xi_n \leq t\} \), and denote \( F_t \), the \( \sigma \)-algebra generated by \( \{B_i, t \geq 0\} \). Then \( (\Omega, P, \{F_t\}) \) is a probability space. Let \( B \) be the Banach space with the norm defined for any \( \psi \in B \),

\[
\mathbb{P} \psi \mathbb{P}^t = (\sup_{t \in [\tau, T]} E \mathbb{P} \psi(t) \mathbb{P}^t),
\]

where \( \psi(t) \), for any given \( t \in [\tau, T] \).
**Definition 5.2.1**

The fractional order integral of the function \( h \in L^1([a,b], \mathbb{R}^n) \) of order \( \alpha \in \mathbb{R}^+ \) is defined by

\[
I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(s)}{(t-s)^{1-\alpha}} ds,
\]

where \( \Gamma(.) \) is the gamma function.

**Definition 5.2.2**

The \( \alpha^{th} \) Riemann-Liouville fractional order of \( h \) on the given interval \([a,b]\) is defined by

\[
(D_a^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{h(s)}{(t-s)^{\alpha+1-n}} ds,
\]

here \( n = \lfloor \alpha \rfloor + 1 \) and \( \lfloor \alpha \rfloor \) denotes the integer part of \( \alpha \).

**Definition 5.2.3**

The Caputo fractional order of \( h \) on the given interval \([a,b]\) is defined by

\[
(^c D_a^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds,
\]

here \( n = \lfloor \alpha \rfloor + 1 \) and \( \lfloor \alpha \rfloor \) denotes the integer part of \( \alpha \).
Lemma 5.2.1 [91]

Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on $[a,b)$ and $s(t)$ is a nonnegative, nondecreasing function defined on $s(t) \leq M$, $t \in [a,b)$, and suppose $z(t)$ is nonnegative and locally integrable on $[a,b)$ with $z(t) \leq a(t) + \int_a^t (t-s)^{\alpha-1} z(s)ds$, $t \in [a,b)$.

Then $z(t) \leq a(t) + \int_0^t \left( \sum_{n=1}^{\infty} \frac{(s(t)\Gamma(n\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} a(s) \right) ds$, $t \in [a,b)$.

If $a(t)$ be a nondecreasing function on $[a,b)$, then

$$z(t) \leq a(t) E_{\alpha} (s(t)\Gamma(\alpha) t^\alpha),$$

where $E_{\alpha}$ is the Mittag-Leffler function $KST$ defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}, \Re(\alpha) > 0.$$

Definition 5.2.4

For a given $T \in (\tau, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B}, \tau \leq t \leq T\}$ is called a solution to the equations (5.2.1) - (5.2.3) in $(\Omega, \mathcal{P}, \{\mathcal{F}_t\})$, if

(i) $x(t) \in \mathcal{B}$ is $\mathcal{F}_t$ - adapted.

(ii) $x(t) = \sum_{k=0}^{\infty} \left( \prod_{i=1}^k b_i(\tau_i) x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^k \prod_{l=1}^j b_j(\tau_j) \int_{\xi_{j-1}}^{\xi_j} (t-s)^{\alpha-1} f(s,x(s)) ds \right)$ (5.2.4)

120
where \( \prod_{j=m}^{n}(\tau) = 1 \) as \( m > n \), \( \prod_{j=i}^{k}b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1}) \cdots b_i(\tau_i) \), and \( I_A(t) \) is the index function, i.e., \( I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \not\in A. \end{cases} \)

### 5.3 Existence and Uniqueness

In this section, the existence and uniqueness of the solution of the system (5.2.1)-(5.2.3) are discussed with the following hypotheses:

**\((H_1)\)**: The function \( f \) satisfies the Lipschitz condition. That is, for \( \vartheta, \hat{\vartheta} \in \mathbb{R}^n \) and \( \tau \leq t \leq T \) there exists a constant \( L > 0 \) such that

\[
E \mathbb{P} f(t, \vartheta) - f(t, \hat{\vartheta}) \mathbb{P} \leq LE \mathbb{P} \vartheta - \hat{\vartheta} \mathbb{P},
\]

\[
E \mathbb{P} f(t, 0) \mathbb{P} \leq k, \text{ where } k \geq 0 \text{ is a constant}.
\]

**\((H_2)\)**: The condition \( \max_{i,k} \prod_{j=1}^{k} P_{b_j}(\tau_{j} P_j) \) is uniformly bounded if there is a constant \( C > 0 \) such that \( \max_{i,k} \prod_{j=1}^{k} P_{b_j}(\tau_{j} P_j) \leq C \) for all \( \tau_j \in D_j, j = 1, 2, \ldots \)

**Theorem 5.3.1**

Let the hypotheses \((H_1)\) and \((H_2)\) hold. If the following inequality

\[
\Lambda = \max\{1, C^2\} \frac{L(T - \tau)^{2\alpha}}{[\alpha \Gamma(\alpha + 1)]} < 1,
\]

is satisfied, then the system (5.2.1)-(5.2.3) has a unique solution in \( B \).
Proof

Let T be an arbitrary number \( \tau < T < +\infty \). Define the nonlinear operator \( S : B \to B \) as follows.

\[
(Sx)(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j)^{\alpha-1} f(s, x(s)) ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds I_{[\xi_k, \xi_{k+1}]}(t), \quad t \in [\tau, T].
\]

It is easy to prove the continuity of \( S \). Choose

\[
r > 4(C^2 P \chi_0 P + \max\{1, C^2\} \frac{(T-\tau)^{2\alpha}}{\alpha[\Gamma(\alpha+1)]}).
\]

Then, to show that \( SB_r \subset B_r \) where \( B_r := \{x \in B \mid P \chi_0 P \leq r\} \). So let

\[
x \in B_r \text{ then}
\]

\[
P(Sx)(t) \leq \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i)p \chi_0 P + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j)p \chi_0 P \int_{\xi_i}^{t} (t-s)^{\alpha-1} P f(s, x(s)) P ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} P f(s, x(s)) P ds I_{[\xi_k, \xi_{k+1}]}(t)\]

\[
\leq 2\sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i)p \chi_0 P I_{[\xi_k, \xi_{k+1}]}(t) \]

\[
+ \int_{\xi_k}^{t} (t-s)^{\alpha-1} P f(s, x(s)) P ds I_{[\xi_k, \xi_{k+1}]}(t)^2
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} P f(s, x(s)) P ds I_{[\xi_k, \xi_{k+1}]}(t)^2
\]
\[ \leq 2 \max \{ \prod_{i=1}^{k} \mathbb{P}(\tau_i) \mathbb{P} \} \mathbb{P} x_0 \mathbb{P} + 2 \max \{ 1, \prod_{i,j=1}^{k} \mathbb{P}(\tau_{ij}) \mathbb{P} \}^2 \]

\[ \left[ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \mathbb{P} f(s,x(s)) \mathbb{P} ds \right] ^{2} \]

\[ \leq 2 \mathbb{C}^2 \mathbb{P} x_0 \mathbb{P} + 2 \max \{ 1, \mathbb{C}^2 \} \left[ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \mathbb{P} f(s,x(s)) - f(s,0) \mathbb{P} + \mathbb{P} f(s,0) \mathbb{P} ds \right] ^{2} \]

\[ E \mathbb{P}(S_x)(t) \mathbb{P} \leq 2 \mathbb{C}^2 \mathbb{P} x_0 \mathbb{P} \]

\[ + 2 \max \{ 1, \mathbb{C}^2 \} \frac{(T-\tau)^{2\alpha}}{\alpha[\Gamma(\alpha+1)]} (Lr + k) \leq r \]

by the choice of \( L \) and \( r \). Now to show that \( S \) is a contraction mapping.

\[ \mathbb{P}(S_x)(t) - (S_y)(t) \mathbb{P} \leq \sum_{k=0}^{w} \sum_{i,j=1}^{k} \mathbb{P}(\tau_{ij}) \mathbb{P} \mathbb{P} \frac{1}{\Gamma(\alpha)} \int_{\xi_{ij}}^{\xi_{ij+1}} (t-s)^{\alpha-1} \mathbb{P} f(s,x(s)) - f(s,y(s)) \mathbb{P} ds \]

\[ + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \mathbb{P} f(s,x(s)) - f(s,y(s)) \mathbb{P} ds \left[ \mathbb{P}(\tau_{ij+1}) \mathbb{P} \right] ^{2} \]

\[ \leq \left[ \max \{ 1, \prod_{i,j=1}^{k} \mathbb{P}(\tau_{ij}) \mathbb{P} \} \right] ^{2} \]

\[ \left[ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \mathbb{P} f(s,x(s)) - f(s,y(s)) \mathbb{P} ds \right] ^{2} \]

123
Taking supremum over $t$,

\[
E \mathbb{P} (S_x(t) - (S_y)(t))^2 \leq \max\{1, C^2\} \frac{(T - \tau)^{\alpha}}{[\Gamma(\alpha + 1)]^\alpha} \int_0^t (t - s)^{\alpha - 1} E \mathbb{P} f(s, x(s)) - f(s, y(s))^2 \mathbb{P} ds
\]

\[
\leq \max\{1, C^2\} \frac{(T - \tau)^{\alpha}}{[\Gamma(\alpha + 1)]^\alpha} L \int_0^t (t - s)^{\alpha - 1} E \mathbb{P} x(s) - y(s)^2 \mathbb{P} ds.
\]

Thus, $E \mathbb{P} (S_x - (S_y))^2 \leq \Lambda E \mathbb{P} x - y^2$.

since $0 < \Lambda < 1$. This shows that the operator $S$ satisfies the contraction principle and therefore, $S$ has a unique fixed point which is the solution of the system (5.2.1) - (5.2.3). This completes proof.

### 5.4 Stability

In this section, the stability of the system (5.2.1) - (5.2.3) through the continuous dependence of solutions on initial conditions is discussed.

**Theorem 5.4.1**

Let $x(t)$ and $\overline{x}(t)$ be solutions of the system (5.2.1) - (5.2.3) with initial values $x_0$ and $\overline{x}_0 \in \mathbb{R}^n$ respectively. If the assumptions of Theorem 5.3.1 are satisfied, then the solution of the system (5.2.1) - (5.2.3) is stable in the mean square.
Proof

By the assumption, $x$ and $\bar{x}$ are the two solutions of the system (5.2.1) - (5.2.3) for $t \in [\tau, T]$. Then,

$$x(t) - \bar{x}(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) [x_0 - \bar{x}_0]$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \left[ \int_{\hat{\xi}_{i-1}}^{\hat{\xi}_i} (t - s)^{\alpha-1} [f(s, x(s)) - f(s, \bar{x}(s))] ds \right]$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{\hat{\xi}_k}^{t} (t - s)^{\alpha-1} [f(s, x(s))) - f(s, \bar{x}(s))] ds] I_{[\hat{\xi}_k, \hat{\xi}_{k+1}]}(t), \quad t \in [\tau, T].$$

By using the hypotheses $(H_1), (H_2)$,

$$E P x(t) - \bar{x}(t) P \leq 2 \left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} P b_i(\tau_i) P \ E P x_0 - \bar{x}_0 P I_{[\hat{\xi}_k, \hat{\xi}_{k+1}]}(t) \right]$$

$$+ 2 \left[ \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) P \left\{ \frac{1}{\Gamma(\alpha)} \int_{\hat{\xi}_{i-1}}^{\hat{\xi}_i} (t - s)^{\alpha-1} P f(s, x(s)) - f(s, \bar{x}(s)) ds \right\} \right]$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{\hat{\xi}_k}^{t} (t - s)^{\alpha-1} P f(s, x(s)) - f(s, \bar{x}(s)) ds \right\} \right]$$

$$\leq 2 \max \{ \prod_{i=1}^{k} P b_i(\tau_i) P \} E P x_0 - \bar{x}_0 P + 2 \left[ \max \{ 1, \prod_{i=1}^{k} b_i(\tau_i) P \} \right]^2$$

$$E \left[ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha-1} P f(s, x(s)) - f(s, \bar{x}(s)) ds \right] I_{[\hat{\xi}_k, \hat{\xi}_{k+1}]}(t) \right]^2$$

125
\[\leq 2C^2E\mathbb{P}x_0 - \bar{x}_0 \mathbb{P} + 2\max\{1, C^2\} \frac{(T - \tau)^\alpha}{[\Gamma(\alpha + 1)]} \int_0^t (t - s)^{\alpha - 1} E\mathbb{P}f(s, x(s)) - f(s, \bar{x}(s))\mathbb{P} \, ds\]

\[\leq 2C^2E\mathbb{P}x_0 - \bar{x}_0 \mathbb{P} + 2\max\{1, C^2\} \frac{(T - \tau)^\alpha}{[\Gamma(\alpha + 1)]} L \int_0^t (t - s)^{\alpha - 1} E\mathbb{P}x(s) - \bar{x}(s)\mathbb{P} \, ds\]

\[
\sup_{t \in [\tau, T]} E\mathbb{P}x(t) - \bar{x}(t)\mathbb{P} \leq 2C^2E\mathbb{P}x_0 - \bar{x}_0 \mathbb{P}
\]

\[+ 2\max\{1, C^2\} \frac{(T - \tau)^\alpha}{[\Gamma(\alpha + 1)]} L \int_0^t (t - s)^{\alpha - 1} \sup_{s \in [\tau, T]} E\mathbb{P}x(s) - \bar{x}(s)\mathbb{P} \, ds\]

\[\mathbb{P}x - \bar{x} \mathbb{P} \leq 2C^2E\mathbb{P}x_0 - \bar{x}_0 \mathbb{P} + 2\max\{1, C^2\} \frac{(T - \tau)^\alpha}{[\Gamma(\alpha + 1)]} L \mathbb{P}x - \bar{x} \mathbb{P} \int_0^t (t - s)^{\alpha - 1} \, ds\]

\[
\leq 2C^2E\mathbb{P}x_0 - \bar{x}_0 \mathbb{P} + 2\max\{1, C^2\} \frac{(T - \tau)^{2\alpha}}{\alpha[\Gamma(\alpha + 1)]} L \mathbb{P}x - \bar{x} \mathbb{P}.
\]

\[\mathbb{P}x - \bar{x} \mathbb{P} \leq \zeta E\mathbb{P}x_0 - \bar{x}_0 \mathbb{P} \text{ where, } \zeta = \frac{2C^2}{1 - 2\max\{1, C^2\} \frac{(T - \tau)^{2\alpha}}{\alpha[\Gamma(\alpha + 1)]} L}.
\]

Now given \(\varepsilon > 0\), choose \(\delta = \frac{\varepsilon}{\zeta}\) such that \(E\mathbb{P}x_0 - \bar{x}_0 \mathbb{P} \leq \delta\).

Then \(\mathbb{P}x - \bar{x} \mathbb{P} \leq \varepsilon\).

Thus, it is apparent that the difference between the solution \(x(t)\) and \(\bar{x}(t)\) in the interval \([\tau, T]\) is small provided the change in the initial point \((t_0, x_0)\) as well as in the function \(f(t, x(t))\) do not exceed prescribed amounts. This completes the proof.
5.5 Ulam-Hyers Stability Results

In this section, the Ulam-Hyers stability of random impulsive fractional differential equation (5.2.1) - (5.2.3) is found out.

Let $\varepsilon > 0$ and $\phi: [\tau, T] \rightarrow \mathbb{R}^+$ be a continuous function.

Consider the following inequalities:

\[
\begin{align*}
\mathbb{P} D_t^\alpha x(t) - f(t, x(t)) & \leq \varepsilon, \quad t \neq \xi_k, \quad t \geq \tau. \\
\mathbb{P} x(\xi_k^+) - b_k(\xi_k^-) x(\xi_k^-) & \leq \varepsilon, \quad k = 1, 2, \ldots, 
\end{align*}
\]  
\hspace{1cm} (5.5.1)

\[
\begin{align*}
\mathbb{P} D_t^\alpha x(t) - f(t, x(t)) & \leq \phi(t), \quad t \neq \xi_k, \quad t \geq \tau. \\
\mathbb{P} x(\xi_k^+) - b_k(\xi_k^-) x(\xi_k^-) & \leq \phi(t), \quad k = 1, 2, \ldots, 
\end{align*}
\]  
\hspace{1cm} (5.5.2)

\[
\begin{align*}
\mathbb{P} D_t^\alpha x(t) - f(t, x(t)) & \leq \varepsilon \phi(t), \quad t \neq \xi_k, \quad t \geq \tau. \\
\mathbb{P} x(\xi_k^+) - b_k(\xi_k^-) x(\xi_k^-) & \leq \varepsilon \phi(t), \quad k = 1, 2, \ldots, 
\end{align*}
\]  
\hspace{1cm} (5.5.3)

**Definition 5.5.1**

The system (5.2.1) - (5.2.3) is Ulam-Hyers stable in the mean square if there exists a real number $\kappa > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in \mathcal{B}$ of the inequality (5.5.1) there exists a solution $y \in \mathcal{B}$ of the system (5.2.1) - (5.2.3) with

\[E \mathbb{P} x(t) - y(t) \leq \kappa \varepsilon, \quad t \in [\tau, T].\]
Definition 5.5.2

The system (5.2.1) - (5.2.3) is generalized Ulam-Hyers stable in the mean square if there exists a real number \( \eta \in \mathbb{B}, \eta(0) = 0 \) such that for each solution \( x \in \mathbb{B} \) of the inequality (5.5.1) there exists a solution \( y \in \mathbb{B} \) of the system (5.2.1) - (5.2.3) with \( E\mathbb{P}x(t) - y(t) \leq \eta(t), \ t \in [\tau, T] \).

Definition 5.5.3

The system (5.2.1) - (5.2.3) is Ulam-Hyers-Rassias stable in the mean square with respect to \( \phi \) if there exists a real number \( \zeta > 0 \) such that for each \( \varepsilon > 0 \) and for each solution \( x \in \mathbb{B} \) of the inequality (5.5.1) there exists a solution \( y \in \mathbb{B} \) of the system (5.2.1) - (5.2.3) with

\[
E\mathbb{P}x(t) - y(t) \leq \zeta \varepsilon \phi(t), \quad t \in [\tau, T].
\]

Definition 5.5.4

The system (5.2.1) - (5.2.3) is generalized Ulam-Hyers-Rassias stable in the mean square with respect to \( \phi \) if there exists a real number \( \zeta > 0 \) such that for each solution \( x \in \mathbb{B} \) of the inequality (5.5.1) there exists a solution \( y \in \mathbb{B} \) of the system (5.2.1) - (5.2.3) with

\[
E\mathbb{P}x(t) - y(t) \leq \zeta \phi(t), \quad t \in [\tau, T].
\]
Remark 5.5.1

It is clear that

1. Definition (5.5.1) \(\Rightarrow\) Definition (5.5.2)

2. Definition (5.5.2) \(\Rightarrow\) Definition (5.5.4)

3. Definition (5.5.3) for \(\phi(t)=1\) \(\Rightarrow\) Definition (5.5.1).

Remark 5.5.2

A function \(x \in \mathbb{B}\) is a solution of the inequality (5.5.1) if and only if there exists a function \(h \in \mathbb{B}\) and the sequence \(h_k, k=1,2,\ldots\) (which depend on \(x\)) such that

1. \(E \mathcal{P}h(t)\mathcal{P} \leq \varepsilon, t \in [\tau, T]\) and \(E \mathcal{P}h_k\mathcal{P} \leq \varepsilon, k=1,2,\ldots\);

2. \(^\alpha\mathcal{D}^\alpha_t x(t) = f(t,x(t)) + h(t), t \neq \xi_k, t \geq \tau\);

3. \(x(\xi_k) = b_k(\tau_k) x(\xi_k^-) + h_k, k=1,2,\ldots\).

One can have similar remarks for the inequalities (5.5.2) and (5.5.3).

Remark 5.5.3

Let \(0 < \alpha < 1\), if \(x \in \mathbb{B}\) is a solution of the inequality (5.5.1) then \(x\) is a solution of the following integral inequality

\[
P x(t) - \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i) x_0 + \frac{1}{\Gamma(\alpha)} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s,x(s))ds
\]
From the remark 5.5.2,

\begin{equation}
\left\{\begin{array}{l}
^cD_t^\alpha x(t) = f(t,x(t)) + h(t), \quad t \neq \xi_k, \quad t \geq \tau. \\
x(\xi_k) = b_k(\tau_k)x(\xi_k^-) + h_k, \quad k = 1,2, \ldots.
\end{array}\right.
\tag{5.5.4}
\end{equation}

Then

\begin{equation}
x(t) = \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i)x_0 + \prod_{i=1}^{k} b_i(\tau_i)h_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s,x(s))ds
\end{equation}

\begin{equation}
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s,x(s))ds + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} h(s)ds
\end{equation}

\begin{equation}
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} h(s)ds I_{[\xi_k,\xi_k^+)}(t), \quad t \in [\tau,T].
\end{equation}

Therefore,

\begin{equation}
E P x(t) - \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s,x(s))ds
\end{equation}

\begin{equation}
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s,x(s))ds I_{[\xi_k,\xi_k^+)}(t)P
\end{equation}

\begin{equation}
= E P \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i)h_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} h(s)ds
\end{equation}
There are similar remarks for the solutions of the inequalities \((5.5.2)\) and \((5.5.3)\).

Now, generalized Ulam-Hyers-Rassias stability results are given in this section.

**Theorem 5.5.1**

Assumption \((H_1)\) and \((H_2)\) hold. Suppose there exists \(\lambda > 0\) such that

\[
\frac{1}{\Gamma(\alpha)} \int_0^\tau (t-s)^{\alpha-1} \phi(s) ds \leq \lambda \phi(t), \text{ for each } t \in [\tau, T],
\]

where \(\phi: [\tau, T] \rightarrow \mathbb{R}^+\) is a continuous nondecreasing function. Then the system \((5.2.1)-(5.2.3)\) is generalized Ulam-Hyers-Rassias stable in the mean square.
Proof

Let $x \in B$ be a solution of the inequality (5.5.2). By Theorem 5.3.1 there exist a unique solution $y$ of the random impulsive fractional differential system

$$
\begin{cases}
\mathcal{D}_t^\alpha y(t) = f(t, y(t)), & t \neq \xi_k, \quad t \geq \tau \\
y(\xi_k) = b_k(\tau_k)y(\xi_k^-), & k = 1, 2, \ldots. \\
y(0) = x_0.
\end{cases}
$$

(5.5.5)

Then

$$
y(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{+\infty} \prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s, y(s))ds
$$

$$
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s, y(s))ds I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [\tau, T].
$$

By differential inequality (5.5.2),

$$
E_p x(t) = \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{+\infty} \prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s, x(s))ds
$$

$$
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s, x(s))ds I_{[\xi_k, \xi_{k+1})}(t) P
$$

$$
\leq 2 \max\left\{ \prod_{i=1}^{k} P b_i(\tau_i) P \right\} E_p P + 2 \max\{1, \prod_{j=1}^{k} P b_j(\tau_j) P \}^2
$$

$$
\times \left[ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \phi(s)ds I_{[\xi_k, \xi_{k+1})}(t) \right]^2
$$

$$
\leq 2 (C + \lambda) \phi(t), \quad t \in [\tau, T].
$$

132
Hence for each \( t \in [\tau, T] \),

\[
E \mathbb{P} x(t) - y(t) \mathbb{P} = E \mathbb{P} x(t) - \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\zeta_{k-1}}^{\zeta_k} (t-s)^{\alpha-1} f(s, y(s)) ds
\]

\[+ \frac{1}{\Gamma(\alpha)} \int_{\zeta_k}^{t} (t-s)^{\alpha-1} f(s, y(s)) ds I_{[\zeta_k, \zeta_{k+1})}(t) \mathbb{P} \]

\[
E \mathbb{P} x(t) - y(t) \mathbb{P} \leq 2E \mathbb{P} x(t) - \sum_{k=0}^{+\infty} \prod_{i=1}^{k} b_i(\tau_i) x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\zeta_{k-1}}^{\zeta_k} (t-s)^{\alpha-1} f(s, x(s)) ds
\]

\[+ \frac{1}{\Gamma(\alpha)} \int_{\zeta_k}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds I_{[\zeta_k, \zeta_{k+1})}(t) \mathbb{P} \]

\[+ 2E \mathbb{P} \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\zeta_{k-1}}^{\zeta_k} (t-s)^{\alpha-1} \{ f(s, x(s)) - f(x, y(s)) \} ds \]

\[+ \frac{1}{\Gamma(\alpha)} \int_{\zeta_k}^{t} (t-s)^{\alpha-1} \{ f(s, x(s)) - f(x, y(s)) \} ds I_{[\zeta_k, \zeta_{k+1})}(t) \mathbb{P} \]

\[\leq 4\{ C^2 + \lambda \} \phi(t) + 2 \max\{1, C^2\} \frac{(T-\tau)^{\alpha}}{[\Gamma(\alpha+1)]} L \int_{0}^{t} (t-s)^{\alpha-1} E \mathbb{P} x(s) - y(s) \mathbb{P} ds\]

By Lemma 5.2.1, there exists a constant \( h > 0 \) independent of \( \lambda \phi(t) \)

Such that

\[E \mathbb{P} x(t) - y(t) \mathbb{P} \leq h 4\{ C^2 + \max\{1, C^2\} \lambda \} \phi(t) = \zeta \phi(t), \quad t \in [\tau, T].\]

Thus, the system (5.2.1)-(5.2.3) is generalized Ulam - Hyers-Rassias stable in the mean square. Hence the proof.
Remark 5.5.4

1. Under the assumption of Theorem 5.5.1, the system (5.2.1)-(5.2.3) and the inequality (5.5.1) are considered. One can repeat the same process to verify that the system (5.2.1)-(5.2.3) is Ulam-Hyers stable in the mean square.

2. Under the assumption of Theorem 5.5.1, the system (5.2.1)-(5.2.3) and the inequality (5.5.3) are considered. One can repeat the same process to verify that the system (5.2.1)-(5.2.3) is Ulam-Hyers-Rassias stable in the mean square.

3. One can extend the above results to case of the system (5.2.1)-(5.2.3) with $T = +\infty$.

5.6 An Example

5.6.1 Example

In this section, an example is given to illustrate the main result. Consider the following random impulsive fractional differential system:

\[
\begin{align*}
^cD_t^\alpha x(t) &= \frac{e^{-yt} |x(t)|}{(1+e^t)(1+|x(t)|)}, \quad \alpha \in (0,1), \quad t \in [0,1] \\
x(\xi_k) &= b_k(\tau_k)x(\xi_k^-), \quad k = 1,2,\ldots, \\
x_0 &= x_0,
\end{align*}
\]

(5.6.1)

where $y > 0$ is a constant.
Fix \( f(t,x) = \frac{e^{-\gamma t} x}{(1+e^t)(1+x)}, \quad (t,x) \in [0,1] \times [0,\infty). \)

Let \( x_1, x_2 \in [0,\infty) \) and \( t \in [0,1] \). Then

\[
\mathcal{P} f(t,x_1) - f(t,x_2) \mathcal{P} \leq \left( \frac{e^{-\gamma t}}{1+e^t} \right)^2 \mathcal{P} \frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} \mathcal{P} \\
\leq \frac{e^{-2\gamma t} \mathcal{P} x_1 - x_2 \mathcal{P}}{(1+e^t)^2 (1+x_1)^2 (1+x_2)^2} \\
\leq \frac{e^{-2\gamma t} \mathcal{P} x_1 - x_2 \mathcal{P}}{(1+e^t)^2} \\
\leq \frac{e^{-2\gamma t} \mathcal{P} x_1 - x_2 \mathcal{P}}{4} \\
\leq L_1 \mathcal{P} x_1 - x_2 \mathcal{P}
\]

and \( \mathcal{P} f(t,0) \mathcal{P} \leq k_1 \) where \( k_1 \geq 0 \) is a constant.

On the other hand it also satisfies the hypothesis \((H_2)\). From all the above facts, in view of Theorem 5.3.1, concluded that the problem (5.6.1) has a unique solution provided

\[
\Lambda_1 = \frac{\max \{ 1, C_1^2 \} L_1}{[\alpha \Gamma(\alpha + 1)]} < 1.
\]

The next results are consequences of Theorems 5.4.1 and 5.5.1 respectively.
**Proposition 5.6.1**

Let the hypotheses \((H_1)\) and \((H_2)\) be hold. Then the solution of the system (5.6.1) is stable in the mean square.

**Proposition 5.6.2**

Let the hypotheses \((H_1)\) and \((H_2)\) be hold. Suppose there exists \(\lambda > 0\) such that

\[
\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) ds \leq \lambda \phi(t), \text{ foreach } t \in [0,1],
\]

where \(\phi:[0,1] \to \mathbb{R}^+\) is a continuous nondecreasing function. Then the system (5.6.1) is generalized Ulam - Hyers- Rassias stable in the mean square.

**5.6.2 Example**

Consider an interest rate model discussed with random impulsive differential equations in [83]. The model to fractional differential equation is generalized. The time when interest rate is adjusted is a random variable. However interest rate is a constant during two neighboring adjusted times. Thus the interest rate \(x(t)\) can be modeled by random impulsive differential equation of the form:
where $0 < \alpha \leq 1$ and $B = \mathcal{R}$. Let \( \{\xi_k\} \) denotes the times that interest rate is adjusted, which are a series of random variable. \( I_k \) is some pending function of \( x(\xi_k) \). When \( \alpha = 1 \) it represents the model in [83].

Let consider the inequality:

\[
\left\{ \begin{array}{l}
\mathbb{P} D^\alpha_t u(t) \leq \phi(t), \quad t \neq \xi_k, \quad t \geq \tau, \\
\mathbb{P} x(\xi_k) - b_k(\tau_k) u(\xi_k^-) \leq \phi(t), \quad k = 1, 2, \ldots,
\end{array} \right.
\]

(5.6.3)

Let \( u \in B \) is a solution of the inequality (5.6.3). Then there exists a function \( h \in B \) such that

\[
(i) \quad E \mathbb{P} h(t) \leq \phi(t), \quad E \mathbb{P} h_k \leq \phi(t) \quad t \in [\tau, T], k = 1, 2, \ldots,
\]

\[
(ii) \quad \left\{ \begin{array}{l}
c D^\alpha_t u(t) = h(t), \quad t \neq \xi_k, \quad t \geq \tau, \\
u(\xi_k) = b_k(\tau_k) u(\xi_k^-) + h_k, \quad k = 1, 2, \ldots,
\end{array} \right.
\]

(5.6.4)

In the equation (5.6.4), integrating

\[
u(t) = \sum_{k=0}^{\infty} \prod_{i=1}^{k} b_i(\tau_i) x_0 + \prod_{i=1}^{k} b_i(\tau_i) h_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_i}^{\xi_i} (t - s)^{\alpha - 1} h(s) ds + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t - s)^{\alpha - 1} h(s) ds \big|_{\xi_k}^{\xi_{k+1}} (t), \quad t \in [\tau, T].
\]

137
Let us take the unique solution \( x(t) \) of \((5.6.2)\) given by

\[
x(t) = \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) x_0 \right] I_{[\xi_k, \xi_{k+1})} (t), \quad t \in [\tau, T].
\]

Then

\[
P u(t) - x(t) P = \mathcal{P} \sum_{k=0}^{+\infty} \left[ \prod_{i=1}^{k} b_i(\tau_i) h_i \right] + \frac{1}{\Gamma(\alpha)} \mathcal{P} \sum_{i=1}^{k} b_i(\tau_i) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} h(s) ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} h(s) ds \mathcal{I}_{[\xi_k, \xi_{k+1})} (t) \mathcal{P}
\]

\[
E \mathcal{P} u(t) - x(t) P \leq \sum_{k=0}^{+\infty} E \mathcal{P} \left[ \prod_{i=1}^{k} b_i(\tau_i) h_i \right] + \frac{1}{\Gamma(\alpha)} E \mathcal{P} \sum_{i=1}^{k} b_i(\tau_i) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} h(s) ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} h(s) ds \mathcal{I}_{[\xi_k, \xi_{k+1})} (t) \mathcal{P}
\]

\[
\leq 2 \mathcal{C}^2 \phi(t) + 2 \max \{1, \mathcal{C}^2\} \left[ \frac{(T-\tau)^{\alpha}}{[\Gamma(\alpha+1)]} \right] \int_{0}^{t} (t-s)^{\alpha-1} \phi(s) ds
\]

If there exists \( \lambda > 0 \) such that

\[
\frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \phi(s) ds \leq \lambda \phi(t), \text{ for each } t \in [\tau, T],
\]

then the equation \((5.6.2)\) is generalized Ulam - Hyers- Rassias stable on \([\tau, T]\) with respect to \( \phi \).
Chapter - 6
6.1 Conclusions

1. Sufficient conditions guaranteeing existence, uniqueness and stability results for nonlinear delay integro-differential equations with random impulses are found out by using Leray-Schauder alternative fixed point theorem and Banach contraction principle.

2. Sufficient conditions guaranteeing existence, uniqueness and stability results for random impulsive semilinear differential equations are found out by using the contraction mapping principle and fixed point theorem mentioned.

3. Sufficient conditions guaranteeing existence, uniqueness and stability results for neutral partial differential equations with random impulses are found out by using Banach fixed point theorem mentioned.

4. Sufficient conditions guaranteeing existence, uniqueness and stability results for random impulsive fractional differential equations are found out by using Hyers-ulam stability and Hyers-Ulam-Rassias stability.


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